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# Periodic Solutions of Nerve Impulse Equations*

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## INTRODUCTION

This paper continues the discussion of singular perturbation solutions of nerve impulse equations begun in [1]. Phase space analysis is used to study a general model of a biological process (e.g., nerve impulse, heartbeat, muscle contraction) consisting of a differential equation coupled with $l$ "slow" and $m$ "fast" equations (Eq. 4.1). The slow [fast] equations correspond to subprocesses whose rates are slow [fast] relative to the rate of the primary phenomenon.

We develop a method of studying the principal, slow, and fast equations separately; piecing together the resulting solutions; and showing that these singular solutions correspond to true solutions of the system for certain parameter values.

The Hodgkin–Huxley [9] equations of nerve impulse transmission consist of a nonlinear diffusion equation coupled with one fast and two slow equations:

$$\begin{aligned} \frac{1}{R}\frac{\partial^2 V}{\partial x^2} &= C\frac{\partial V}{\partial s} + g(V, m, n, h), \\ \frac{\partial m}{\partial s} &= \delta^{-1}\gamma_m(V)\,(m_\infty(V) - m), \\ \frac{\partial n}{\partial s} &= \epsilon\gamma_n(V)\,(n_\infty(V) - n), \\ \frac{\partial h}{\partial s} &= \epsilon\gamma_h(V)\,(h_\infty(V) - h), \end{aligned} \qquad (0.1, \text{HH})$$

where $x$ is the distance from the stimulus and $s$ is the time since the stimulus. $V = V(x, s)$ represents the displacement from rest of the potential difference (in $mV$) across the axon membrane. By Ohm's law, $1/R(\partial^2 V/\partial x^2)$ is the total current across the membrane, where $R$ is the resistance of the axoplasm. This

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current consists of capacitance ($C(\partial V/\partial s)$) and ionic ($g(V, m, n, h)$) currents. The tendency for the membrane to conduct ions is represented by $m$ ($Na^+$ activation, inward), $n$ ($K^+$ activation, outward), and $h$ ($Na^+$ inactivation). In the original Hodgkin–Huxley model of impulses in the giant squid axon, $g(V, m, n, h) = \bar{g}_{Na}m^3h(V - V_{Na}) + \bar{g}_K n^4(V - V_K) + \bar{g}_L(V - V_L)$, where $\bar{g}_{Na}, V_{Na}, \bar{g}_K, V_K, \bar{g}_L, V_L$ are constants, and $\gamma_m, m_\infty, \gamma_n, n_\infty, \gamma_h, h_\infty$ are experimentally determined. We embed this particular model in a large class of models which may be used to represent nerve transmission of various species, as well as other processes, depending upon the choice of functions. Hypothesis (3.1, CUBIC, H) on (0.1, HH) is general and qualitative, containing assumptions such as:

(D) $\partial g/\partial n(V, m_\infty(V), n, h) > 0$ and $\partial g/\partial h(V, m_\infty(V), n, h) < 0$ if $V_K < V < V_{Na}$ and $0 < n, h < 1$;

(F) $n_\infty' > 0$ and $h_\infty' < 0$; and

(G) $\gamma_m, \gamma_n, \gamma_h > 0$.

We do not assume, for example, that $g$ is linear in $V$ or that $Na^+$ and $K^+$ act independently. (See [10, Sections 33–34; 14, Chap. 10] for discussions of this model.)

Since an impulse proceeds along the axon in a wave-like manner, we consider traveling wave solutions of (0.1, HH) (Section 2 and Eq. (4.3, HH)). A single pulse or multiple wave train impulse corresponds to a homoclinic solution of (4.3, HH), that is, a solution which goes to the unique rest point as $t \to \pm\infty$. A self-sustaining oscillatory impulse corresponds to a periodic solution of (4.3, HH). In [1] we give general conditions for the existence of homoclinic and finite wave train solutions of (4.1). In addition we show that periodic solutions exist *provided* $l = 1$ (i.e., there is only a slow variable). These results also apply to the case $l \geqslant 2$ (e.g., the Hodgkin–Huxley equations) if one slow variable is much slower than the other (e.g., if $\gamma_n/\gamma_h \ll 1$). Periodic solutions then oscillate between two plateaus [7, 15]. In this paper we show that the notion of an $l$-dimensional singular solution allows us to prove existence of periodic solutions with $l \geqslant 2$.

In Section 1 the results are presented abstractly, in terms of isolating blocks [4, 16] for an autonomous system. The principal property of a block is that the map which sends a point $u$ in the block to the first point of $u \cdot [0, \infty)$ in its exit set is continuous where defined (Fig. 1). Hypothesis (1.3, PER) gives a sufficient condition for the existence of a periodic solution of an arbitrary autonomous system.

In Section 2 we examine traveling wave solutions of a nonlinear diffusion equation coupled with $l$ slow equations (2.2). Hypothesis (2.1, CUBIC) requires that $G(V, y)$ be a "cubic" function of $V$ for fixed $y$ (Fig. 4); and (2.3, PER, $\tilde{\theta}$) requires the existence of an $l$-dimensional singular solution of (2.2, $\tilde{\theta}$, $\epsilon$). These two hypotheses imply the existence of a periodic solution of $(2.2, \tilde{\theta}, \epsilon)$ for small $\epsilon > 0$.

Periodic traveling wave solutions of the Hodgkin–Huxley equations (0.1, HH) are discussed in Sections 3 and 4. Hypothesis (3.1, CUBIC, H) implies (2.1, CUBIC) with $G(V, n, h) \equiv g(V, m_\infty(V), n, h)$. When $l = 2$ a weak assumption, requiring only piecewise continuity of certain maps, replaces the continuity assumption of $(2.3, \text{PER}, \tilde{\theta})$. The piecewise continuity assumption is satisfied, for example, if the system is analytic. In this case, the Hodgkin–Huxley equations with $m = m_\infty(V)$ admit periodic solutions if $\epsilon$ is small and $m = m_\infty(V)$.

In Section 4, we show that results of Sections 2 and 3 are valid for a system with fast variables whenever an associated system (setting $\delta = 0$) satisfies the appropriate hypotheses and $\delta$ is small. In particular results of Section 3 hold for the full Hodgkin–Huxley equations if $\delta$ is small.

Proofs are contained in Section 5.

The FitzHugh–Nagumo model [6, 11] of nerve impulse transmission consists of a nonlinear diffusion equation with a "cubic" term coupled with *one* slow equation ($l = 1$). Periodic solutions of this system are studied in [1, 3, 8, 12].

I wish to thank Professor Charles Conley for his supervision of the thesis research which directly preceded the work presented here.

*Notation*

$\mathbb{R}^n \equiv$ Euclidean $n$-space,

$u : v \equiv \sum_{i=1}^{n} u_i v_i$ (the usual inner product on $\mathbb{R}^n$) where $u = \langle u_1 \cdots u_n \rangle$, $v = \langle v_1 \cdots v_n \rangle \in \mathbb{R}^n$,

$\partial A$ boundary of $A$,

$\text{cl}(A)$ closure of $A$,

$\text{int}(A)$ interior of $A$,

$\text{dom}(f)$ domain of $f$,

$f \mid A$ $f$ restricted to $A$.

## 1. Periodic Solutions of an Autonomous System

In this section we state a sufficient condition for the existence of a periodic solution of the system:

$$\dot{u} = G(u), \tag{1.1}$$

where $G \in C^1$ and $u \in \Omega \subset \mathbb{R}^n$.



*Definitions*

Let $u \cdot t$ denote a solution of (1.1) for $t \in J$, a subinterval of $\mathbb{R}$. If $K \subseteq J$, let $u \cdot K \equiv \{u \cdot t : t \in K\}$; and let $u \cdot J$ be the *trajectory* containing $u$. $u \cdot \mathbb{R}$ is a *periodic solution* if $u = u \cdot t$ for some $t \neq 0$. $u \cdot K$ is a *positive* (*negative*) *half solution* if $K = [0, \infty)$ $[K = (-\infty, 0)]$.

$\bar{u}$ is a *rest point* of (1.1) if $G(\bar{u}) = 0$. If $\bar{u}$ is a rest point whose eigenvalues have $j$ positive and $(n - j)$ negative real parts, $U(\bar{u}) \equiv \{u \in \Omega : u \cdot t \to \bar{u} \text{ as } t \to -\infty\}$ is a $j$-manifold (the *unstable manifold* of $\bar{u}$); and $S(\bar{u}) \equiv \{u \in \Omega : u \cdot t \to \bar{u}$ as $t \to \infty\}$ is an $(n - j)$-manifold (the *stable manifold* of $\bar{u}$) [3, Chap. 13].

$B$ is a *block* for (1.1) if there exist $C^1$ functions $f_1 \cdots f_N : \mathbb{R}^n \to \mathbb{R}$ such that $B \equiv \bigcap_{i=1}^{N} f_i^{-1}([0, \infty))$ is homeomorphic to $[0, 1]^n$ and $\dot{f}_i \equiv \nabla f_i : G \neq 0$ on $\partial B$. $b^+$ (the *entrance set*) $\equiv \{u \in \partial B : f_i(u) = 0$ and $\dot{f}_i(u) > 0$ for some $i\}$. $b^-$ (the *exit set*) $\equiv u \in \partial B : f_i(u) = 0$ and $\dot{f}_i(u) < 0$ for some $i\}$.

This definition of block is more limited than usual [4]; the $C^1$ isolating-block-with-corners of [16] could be used instead. Defined in this way, a block $B$ for (1.1) has the property that if $| G - \tilde{G} |$ is small, then $B$ is a block for the system:

$$\dot{u} = \tilde{G}(u)$$

with $b^\pm$ remaining invariant.

Example 1.1. A saddle point

$$\begin{aligned} \dot{V} &= W, \\ \dot{W} &= \theta W + F(V). \end{aligned} \tag{1.2}$$

Assume $F \in C^1$ and $\theta \geqslant 0$. Suppose $F(V_0) = 0$ and $F'(V_0) > 0$. If $c > 0$ is small, $B_c \equiv \{\langle V, W \rangle : | W \pm (\theta + 1)(V - V_0)| \leqslant (\theta + 1)c\}$ is a block for $(1.2, \theta)$. Moreover, $b_c^\pm = \{\langle V, W \rangle \in B_c : | W \mp (\theta + 1)(V - V_0)| = (\theta + 1)c\}$; and $u \in S(\langle V_0, 0 \rangle)$ if $u \cdot [0, \infty) \subseteq B_c$.

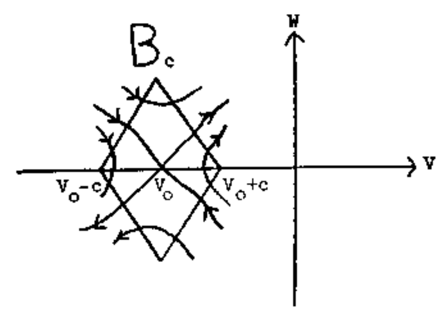


Fig. 1. $B_c$ is a block for $(1.2, \theta)$ if $c$ is small.

$T^{\pm}(u)$ (the time needed for $u$ to reach $b^{\pm}$) $\equiv$

$$\begin{cases} 0, & \text{if } u \in b^{\pm}, \\ \sup\{t > 0 : u \cdot (0, t) \cap b^{\pm} = \phi\}, & \text{if } u \notin b^{\pm}. \end{cases}$$

$\phi^{\pm}(u) \equiv u \cdot T^{\pm}(u)$ if $0 \leqslant T^{\pm}(u) < \infty$. $\phi^{\pm}(u)$ is the first point of $u \cdot [0, \infty)$ in $b^{\pm}$.

$D^{\pm} = \{u : 0 < T^{\pm}(u) < \infty$ and $\phi^{\pm}(u) \notin b^{\mp}\}$. $D^{\pm}$ contains the set on which $\phi^{\pm}$ is defined and continuous. If $\bar{u} \in D^{\pm}$, $\bar{u}$ crosses $\{f_i(u) = 0\}$ transversely at $\phi^{\pm}(\bar{u})$ for some $i$.

LEMMA 1.2. Continuity of maps defined by a block.

*If $B$ is a block, $T^{\pm}$ and $\phi^{\pm}$ are continuous on $D^{\pm}$.*

HYPOTHESIS (1.3, PER). *There exist $B_1$, $B_2$, disjoint blocks for* (1.1), *with properties* (A)–(C).

(A) *No positive half solution is contained in $B_1$ or $B_2$.*

(B) *There exist $\Gamma \subseteq b_1^- \cap D_2^+$, $\Delta \subseteq b_2^- \cap D_1^+$ such that $(b_1^- - \Gamma)$ consists of two components, $\alpha_0$ and $\alpha_1$; and $(b_2^- - \Delta)$ consists of two components, $\beta_0$ and $\beta_1$. In addition, if $\gamma_i \equiv \alpha_i \cap \mathrm{cl}(\Gamma)$ and $\delta_i \equiv \beta_i \cap \mathrm{cl}(\Delta)$, then $\phi_2^- \circ \phi_2^+(\gamma_i) \subseteq \mathrm{int}(\beta_i)$ and $\phi_1^- \circ \phi_1^+(\delta_i) \subseteq \mathrm{int}(\alpha_i)$ $(i = 0, 1)$.*

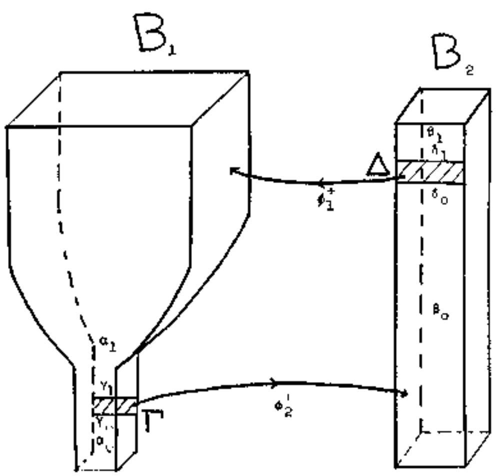


FIG. 2. $B_1$, $B_2$ satisfy Hypothesis (1.3, PER), with $b_1^-$ = front, back, and bottom; $b_2^-$ = front, back, and top.



(C) *There exist homeomorphisms $h_j : b_j^- \to [0, 1]^{n-2} \times [-1, 2]$ such that*

$$h_1(\Gamma) = [0, 1]^{n-2} \times (0, 1);$$
$$h_1(\gamma_i) = [0, 1]^{n-2} \times \{i\};$$
$$h_2(\Delta) = [0, 1]^{n-2} \times (0, 1);$$

*and*

$$h_2(\delta_i) = [0, 1]^{n-2} \times \{i\}.$$

THEOREM 1.4. Periodic solutions of (1.1). *Hypothesis* (1.3, PER) *implies that* (1.1) *admits a periodic solution.*

The proof of (1.4) appears in [1]. It involves the fact that a map $f$ (essentially $\phi_1^- \circ \phi_1^+ \circ \phi_2^- \circ \phi_2^+$) from $\Gamma$ into $b_1^-$ has nonzero degree [5, 13] and hence a fixed point. This fixed point is contained in a periodic solution of (1.1).

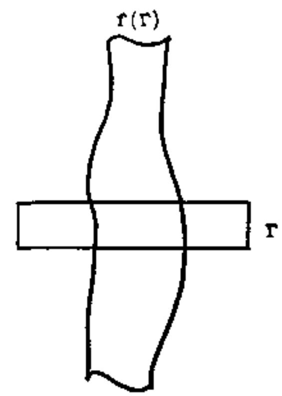


FIG. 3. $\deg(f - I, \Gamma, 0) = \pm 1$, so $f$ has a fixed point.

## 2. PERIODIC TRAVELING WAVE SOLUTIONS OF A NONLINEAR DIFFUSION EQUATION

In this section we apply the abstract results of Section 1 to find traveling wave solutions of a nonlinear diffusion equation coupled with $l$ "slow" equations:

$$\begin{aligned} \partial^2 \bar{V}/\partial x^2 &= \partial \bar{V}/\partial s + G(\bar{V}, \bar{y}), \\ \partial y/\partial s &= \epsilon H(\bar{V}, y), \end{aligned} \tag{2.1}$$

where $V \in (V_\alpha, V_\beta) \subset \mathbb{R}$; $y \in \Omega_y \subset \mathbb{R}^l$; $\epsilon > 0$, and $\Omega_y$ is homeomorphic to $[0, 1]^l$.

A traveling wave solution of (2.1) (with speed $\theta$) consists of a solution $\langle \bar{V}(x, s), \bar{y}(x, s)\rangle$ of (2.1) such that $\bar{V}(x, s) = V(t)$, $y(x, s) = y(t)$ where $t = (x + \theta s)$. An application of the chain rule gives traveling wave solutions of (2.1) as solutions $\langle V, W, y\rangle$ of the $(2 + l)$ equations:

$$\begin{aligned} \dot{V} &= W, \\ \dot{W} &= \theta W + G(V, y), \\ \dot{y} &= \epsilon\theta^{-1}H(V, y). \end{aligned} \tag{2.2}$$

Hypothesis (2.1, CUBIC) will provide the principal restriction on $G$, namely, that $G$ be a "cubic" function of $V$ for fixed $y$ (Fig. 4).

HYPOTHESIS (2.1, CUBIC). (A) $V_\alpha < 0 < V_\beta$, *and* $G(V_\alpha, y) < 0 < G(V_\beta, y)$ *for every* $y$.

(B) *For every* $y$ *there exist at most* three $V$ *such that* $G(V, y) = 0$; *for some* $y$, *there exist exactly three. Moreover,* $\partial^2 G/\partial V^2(V, y) \neq 0$ *if* $G(V, y) = \partial G/\partial V(V, y) = 0$.

(C) $\partial G/\partial y_k > 0$ *for some* $k$.

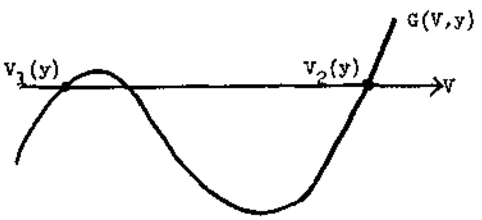


FIG. 4. "Cubic" $G$.

*Definitions* $(\chi, \chi_i, \Pi_i, V_i, y \cdot^i t, \Lambda(y, \theta))$

Let $\chi = \{\langle V, W, y\rangle : W = G(V, y) = 0$ and $\partial G/\partial V(V, y) > 0\}$. Hypothesis (2.1, CUBIC) then implies that $\chi$ has two components, $\chi_1$ and $\chi_2$, with $V < V'$ when $\langle V, 0, y\rangle \in \chi_1$ and $\langle V', 0, y\rangle \in \chi_2$. For $i = 1, 2$, let $\Pi_i$ be the image of $\chi_i$ under the map $\langle V, W, y\rangle \to y$. By the implicit function theorem, there exists $V_i: \Pi_i \to (V_\alpha, V_\beta)$ such that $\langle V, 0, y\rangle \in \chi_i$ iff $y \in \Pi_i$ and $V = V_i(y)$. Note that $V_1(y) < V_2(y)$ when $y \in \Pi_1 \cap \Pi_2$; $\partial V_i/\partial y_j = -\partial G/\partial y_j/\partial G/\partial V$; and sgn $(\partial V_i/\partial y_j) = -\text{sgn}(\partial G/\partial y_j)$.

For $y \in \Pi_i$, (2.2) defines a system on $\Pi_i$:

$$\dot{y}' = H(V_i(y), y). \tag{2.3$_i$}$$

(See Fig. 13, where $l = 2$.) Let $y \cdot^i t$ denote a solution of $(2.3)_i$.

For every $y \in \Pi_1 \cap \Pi_2$ and $\theta \in \mathbb{R}$, $\langle V_i(y), 0, y\rangle$ is a rest point of $(2.2, |\theta|, 0)$ with one positive and one negative eigenvalue. If $\theta > 0$ let $\Lambda(y, \theta)$ be that



branch of $U(\langle V_1(y), 0, y\rangle)$ with a negative half solution contained in $\{W > 0\}$; and let $\Lambda(y, -\theta)$ be that branch of $U(\langle V_2(y), 0, y\rangle)$ with a negative half solution contained in $\{W < 0\}$ (Fig. 5).

Lemma 2.2 shows that for fixed $y \in \Pi_1 \cap \Pi_2$ $\Lambda(y, \theta)$ or $\Lambda(y; -\theta)$ runs between $\chi_1$ and $\chi_2$ in $(2.2, \theta, 0)$ for some $\theta \geqslant 0$.

LEMMA 2.2. Jump Sets Exist. *Assume Hypothesis* (2.1, CUBIC).

(A) *There exists a continuous function* $\theta(y): \Pi_1 \cap \Pi_2 \to \mathbb{R}$ *such that* $\Lambda(y, \theta(y))$ *is a solution of* $(2.2, |\theta(y)|, 0)$ *from* $\langle V_1(y), 0, y\rangle$ *to* $\langle V_2(y), 0, y\rangle$ *if*

$$\int_{V_1(y)}^{V_2(y)} G(V, y)\, dV \leqslant 0 \qquad (\theta(y) \geqslant 0);$$

*or from* $\langle V_2(y), 0, y\rangle$ *to* $\langle V_1(y), 0, y\rangle$ *if*

$$\int_{V_1(y)}^{V_2(y)} G(V, y)\, dV \geqslant 0 \qquad (\theta(y) \leqslant 0).$$

(B) $\theta(y)$ *decreases as* $y_k$ *increases, where* $\partial G/\partial y_k > 0$.

(C) $\{y \in \Pi_1 \cap \Pi_2 : \theta(y) = \tilde{\theta}\}$ *is an* $(l - 1)$*-manifold. If* $\tilde{\theta} \geqslant 0$, $\{\theta(y) = \tilde{\theta}\}$ *is the jump set from* $\Pi_1$ *to* $\Pi_2$; *if* $\tilde{\theta} \leqslant 0$ *it is the jump set from* $\Pi_2$ *to* $\Pi_1$ (Fig. 12).

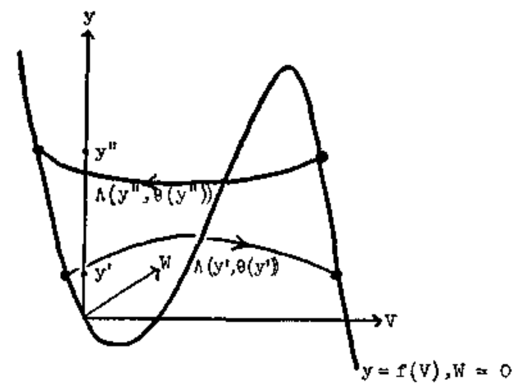


FIG. 5. $l = 1$ and $G(V, y) = -f(V) + y$ (the FitzHugh–Nagumo equations). $\theta(y') = -\theta(y'') > 0$.

We now fix $\tilde{\theta} > 0$ and give a sufficient condition for the existence of a periodic solution of $(2.2, \tilde{\theta}, \epsilon)$ for all small $\epsilon > 0$.

If $y \in \Pi_1$, let $F_1(y)$ be the first point (if any) on $y \cdot^1 [0, \infty) \cap \{\theta(y) = \tilde{\theta}\}$. If $y \in \Pi_2$, let $F_2(y)$ be the first point (if any) on $y \cdot^2 [0, \infty) \cap \{\theta(y) = -\tilde{\theta}\}$.

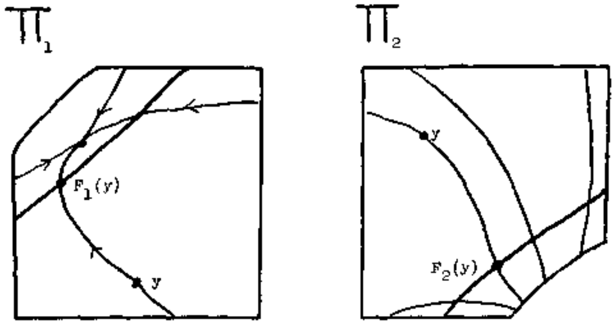


FIG. 6. $F_i$ and $F_2$ with $l = 2$.

Note that $F_i$ is continuous in a neighborhood of $y$ if $\dot\theta^i(F_i(y)) \neq 0$. Let $T_i(y)$ (the time for $y$ to reach $\{\theta(y) = \tilde\theta\}$ or $\{\theta(y) = -\tilde\theta\}$) be the smallest $t \geq 0$ (if any) such that:

$$y \overset{i}{\cdot} T_i(y) = F_i(y).$$

Our theorems remain true if $F_1, F_2$ are replaced by the more general $\tilde F_1, \tilde F_2$, where $\tilde F_1(y)$ is the first point (if any) on $y \cdot^1 [0, \infty) \cap \{y : \theta(y) = \tilde\theta$ or $\partial G/\partial V(V_1(y), y) = 0\}$; and $\tilde F_2(y)$ is the first point (if any) on $y \cdot^2 [0, \infty) \cap \{y : \theta(y) = -\tilde\theta$ or $\partial G/\partial V(V_2(y), y) = 0\}$. In this case, $\{\theta(y) = \tilde\theta\}$ is defined to be $\{y \in \Pi_1 \cap \Pi_2 : \theta(y) = \tilde\theta\} \cup \{y \in \partial\Pi_1 : \operatorname{Lim}_{\tilde y \to y} \theta(\tilde y) \geq \tilde\theta\}$. $\{\theta(y) = -\tilde\theta\}$ is defined analogously.

Hypothesis (2.3, PER, $\tilde\theta$) implies the existence of an *l-dimensional singular solution* of (2.2, $\tilde\theta$). The set $M$ is contained in the jump set from $\Pi_1$ to $\Pi_2$. Let $\tilde M = \{\langle V_1(y), 0, y\rangle : y \in M\}$. $\tilde M$ jumps from $\chi_1$ to $\chi_2$ when $\theta = \tilde\theta$, $\epsilon = 0$. It then moves along $\chi_2$ (in the flow $(2.3)_2$) until it reaches the jump set from $\chi_2$ to $\chi_1$, via the map $F_2$. $\tilde M$ jumps back to $\chi_1$ and returns interior to itself (by (C)) via the map $F_1$ (Fig. 7). Note that $F_1 \circ F_2$ has a fixed point in $M$. (See Fig. 5 for the case $l = 1$.) Hypothesis (2.3) (D) simplifies the proof.

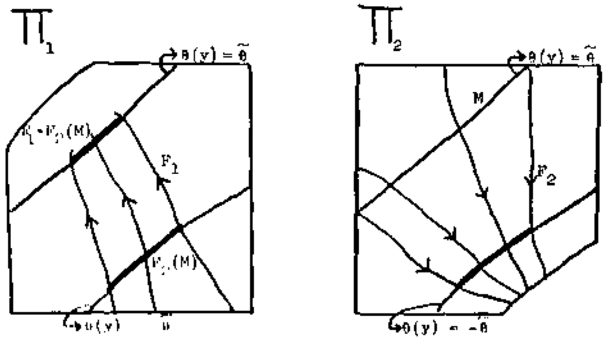


FIG. 7. $M \subseteq \{\theta(y) = \tilde\theta\}$.



HYPOTHESIS (2.3, PER, $\tilde\theta$). *There exists* $M \subseteq \{\theta(y) = \tilde\theta\}$ *such that:*

(A) $M$ *is homeomorphic to* $[0, 1]^{l-1}$;

(B) $F_2$ *is defined and continuous on* $M$ *and* $F_1$ *is defined and continuous on* $F_2(M)$;

(C) $F_1 \circ F_2(M) \subseteq \operatorname{int}(M)$;

(D) $\dot\theta^1 > 0$ *on* $\{\theta(y) = -\tilde\theta\}$ *and* $\dot\theta^2 < 0$ *on* $\{\theta(y) = \tilde\theta\}$.

THEOREM 2.4. Periodic solutions of (2.2, $\tilde\theta$, $\epsilon$). *Assume hypotheses* (2.1, CUBIC) *and* (2.3, PER, $\tilde\theta$). *Then* (2.2, $\tilde\theta$, $\epsilon$) *admits a periodic solution for all small* $\epsilon > 0$.

*Remarks.* $l = 1$ vs $l \geq 2$. 1. Hypothesis (2.3, PER, $\tilde\theta$) (C) implies that $l \geq 2$. In [1] we show that (2.2, $\tilde\theta$, $\epsilon$) admits a periodic solution when $l = 1$ if $H(V_1(y), y) < 0 < H(V_2(y), y)$ whenever $\theta(y) \in [-\tilde\theta, \tilde\theta]$ (Fig. 5). In particular, the FitzHugh–Nagumo equations [7, 11] satisfy this condition. If $l = 1$, the hypothesis cannot be satisfied if $H(V_1(\bar y), \bar y) = 0$ (i.e., $\langle V_1(\bar y), 0, \bar y\rangle$ is a rest point of (2.2, $\tilde\theta$, $\epsilon$)) and $\tilde\theta \geq \theta(\bar y)$, contrasting with the fact that Hypothesis (2.2, PER, $\tilde\theta$) could be satisfied if $l \geq 2$ and $\tilde\theta \geq \theta(\bar y)$.

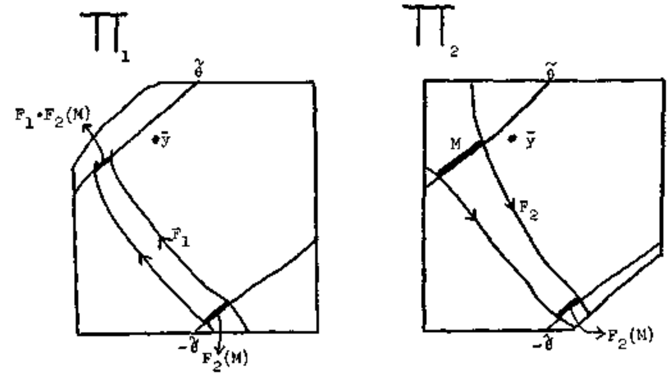


FIG. 8. Hypothesis (2.3, PER, $\tilde\theta$) with $\tilde\theta > \theta(\bar y)$, $l = 2$.

2. The proof of Theorem 2.4 implies that Hypothesis (2.2, PER, $\tilde\theta$) (B) and (C) could be replaced by:

(B)′ $F_2$ is defined and continuous on $(F_1 \circ F_2)^j(M)$ and $F_1$ is defined and continuous on $F_2 \circ (F_1 \circ F_2)^j(M)$ ($j = 0 \cdots (k-1)$ for some $k \geq 1$).

(C)′ $(F_1 \circ F_2)^k(M) \subseteq \operatorname{int}(M)$.

(B)′ and (C)′ are weaker than (2.3, PER, $\tilde\theta$) (B) and (C) iff $l \geq 3$.

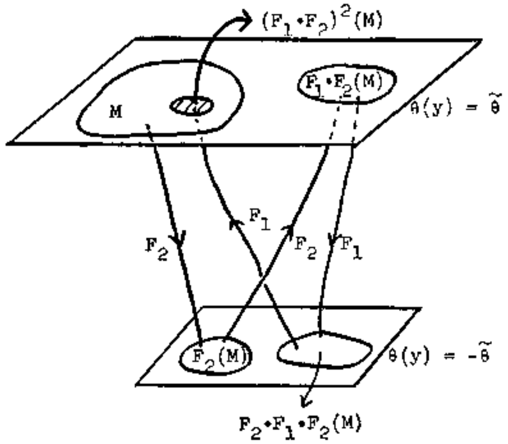


FIG. 9. $(B')$ and $(C')$ with $l = 3$ and $k = 2$.

## 3. Periodic Solutions of the Hodgkin–Huxley Equations with $m = m_\infty(V)$

In this section we consider the system:

$$\begin{aligned} \dot{V} &= W, \\ \dot{W} &= \theta W + G(V, n, h), \\ \dot{n} &= \epsilon\theta^{-1}\gamma_n(V)\,(n_\infty(V) - n), \\ \dot{h} &= \epsilon\theta^{-1}\gamma_h(V)\,(h_\infty(V) - h). \end{aligned} \tag{3.1, H}$$

Let $n_\infty(0) \equiv n_0$, $h_\infty(0) \equiv h_0$.

Hypothesis (3.1, CUBIC, H) will imply Hypothesis (2.1, CUBIC). Since $l = 2$, Hypothesis (2.3, PER, $\tilde{\theta}$) may be replaced by the weaker hypothesis (3.2, PER, $\tilde{\theta}$, H).

HYPOTHESIS (3.1, CUBIC, H). *There exist $V_K < 0 < V_{Na}$ such that for every $V \in [V_K, V_{Na}]$ and $n, h \in [0, 1]$:*

(A) *$G(V_K, n, h) < 0 < G(V_{Na}, n, h)$ and $G \in C^2$.*

(B) *There exist at most three $V \in (V_K, V_{Na})$ such that $G(V, n, h) = 0$. Moreover, if $G(V, n, h) = \partial G/\partial V(V, n, h) = 0$, $\partial^2 G/\partial V^2(V, n, h) \neq 0$.*

(C) *$\partial G/\partial V(0, n_0, h_0) < 0$, and there exists $V_2 > 0$ such that $G(V_2, n_0, h_0) = 0$ and $\int_0^{V_2} G(V, n_0, h_0)\,dV < 0$.*



(D) *$\partial G/\partial n > 0$ and $\partial G/\partial h < 0$.*

(E) *$G(V, n_\infty(V), h_\infty(V)) = 0$ iff $V = 0$.*

(F) *$0 < n_\infty, h_\infty < 1$; $n_\infty, h_\infty \in C^1$; $n_\infty' > 0$; and $h_\infty' < 0$.*

(G) *$\gamma_n, \gamma_h > 0$.*

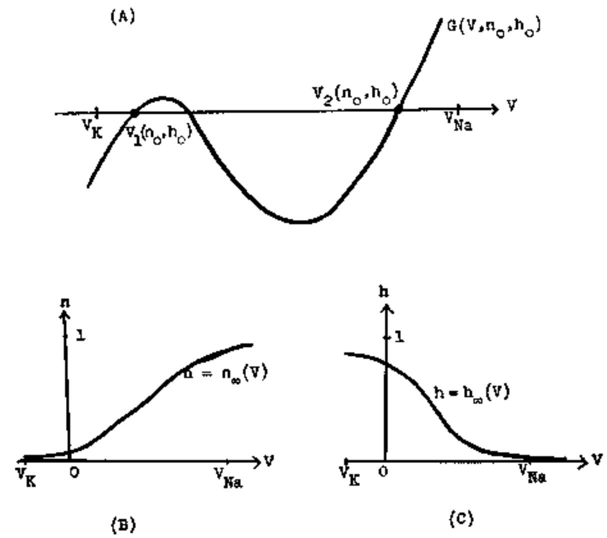


FIG. 10. (A) "Cubic" $G(V, n_0, h_0)$. (B) $n = n_\infty(V)$. (C) $h = h_\infty(V)$.

(3.1, CUBIC, H) implies that $\langle 0, 0, n_0, h_0 \rangle$ is the unique rest point of (3.1, H).

For $i = 1, 2$, there exist functions $V_i(n, h)$ defined on $\Pi_i \subseteq [0, 1]^2$ with $G(V_i(n, h), n, h) = 0$ and $\partial G/\partial V(V_i(n, h), n, h) > 0$. Let $(H)_i$ be the system:

$$\begin{aligned} \dot{n}^i &= \gamma_n(V_i(n, h))\,(n_\infty(V_i(n, h)) - n), \\ \dot{h}^i &= \gamma_h(V_i(n, h))\,(h_\infty(V_i(n, h)) - h), \end{aligned} \tag{3.2, H$_i$}$$

where $\langle n, h \rangle \in \mathrm{cl}(\Pi_i)$.

By Jump Set Lemma (2.2), there exists a function $\theta(n, h)\colon \Pi_1 \cap \Pi_2 \to (-\infty, \infty)$ such that $(H, |\theta(n, h)|, 0)$ admits a solution from $\langle V_1(n, h), 0, n, h \rangle$ to $\langle V_2(n, h), 0, n, h \rangle$ if $\theta(n, h) \geqslant 0$ or from $\langle V_2(n, h), 0, n, h \rangle$ to $\langle V_1(n, h), 0, n, h \rangle$ if $\theta(n, h) \leqslant 0$. Moreover $\partial\theta/\partial n < 0$ and $\partial\theta/\partial h > 0$.

HYPOTHESIS (3.2, PER, $\tilde{\theta}$, H). (A) *$\dot{\theta}^1 > 0$ on $\{\theta(n, h) = -\tilde{\theta}\}$ and $\dot{\theta}^2 < 0$ on $\{\theta(n, h) = \tilde{\theta}\}$.*

(B) *$F_2$ is defined and piecewise-continuous on $\{\theta(n, h) = \tilde{\theta}\}$ and $F_1$ is defined and piecewise-continuous on $F_2(\{\theta(n, h) = \tilde{\theta}\})$.*

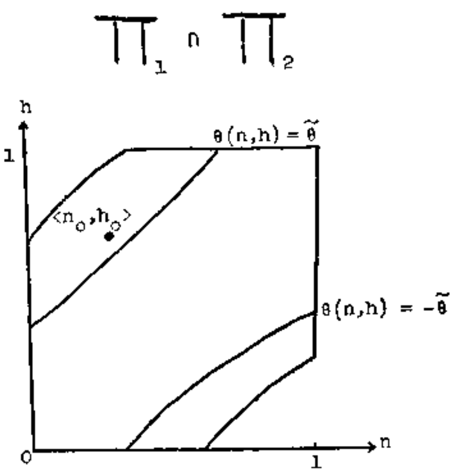


FIG. 11. Jump sets of (3.1, H).

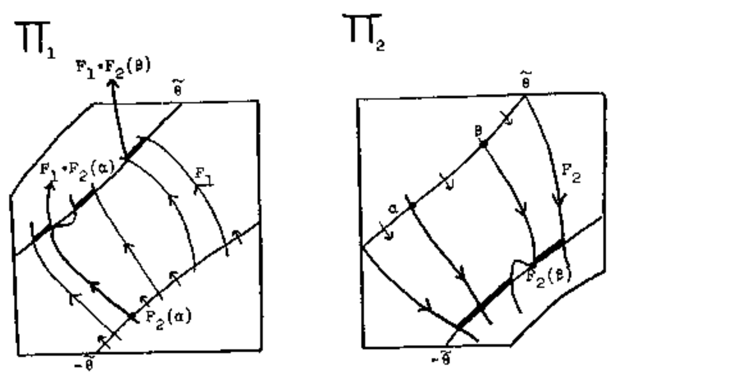


FIG. 12. Hypothesis (3.2, PER, $\tilde{\theta}$, H). $F_1$ is discontinuous at $\beta$; $F_1 \circ F_2$ is discontinuous at $\alpha$ and $\beta$.

*Remarks.* $\tilde{F}_2$ is always defined on $\{\theta(n, h) \geqslant 0\}$ and $\tilde{F}_1$ is always defined on the image of $\tilde{F}_2$ if $\theta(n_0, h_0) > \tilde{\theta}$. Thus (B) may be weakened as remarked previously. (3.2) (A) says that all solutions of $(\mathrm{H})_1$, $(\mathrm{H})_2$ cross $\{\theta(n, h) = -\tilde{\theta}\}$, $\{\theta(n, h) = \tilde{\theta}\}$ in one direction. Since $\dot{\theta}^i = \theta_n \dot{n}^i + \theta_h \dot{h}^i$, (A) is satisfied if $\dot{n}^1 < 0$ and $\dot{h}^1 < 0$ on $\{\theta(n, h) = -\tilde{\theta}\}$; and $\dot{n}^2 < 0$ and $\dot{h}^2 < 0$ on $\{\theta(n, h) = \tilde{\theta}\}$, independent of $\gamma_n, \gamma_h$ (Fig. 13).

Since $F_1[F_2]$ is discontinuous at $\langle n, h\rangle$ only if $\dot{\theta}^1(F_1(n, h)) = 0$, $[\dot{\theta}^2(F_2(n, h)) = 0]$, (3.2, PER, $\tilde{\theta}$) (B) is satisfied if $\{\langle n, h\rangle: \dot{\theta}^1(n, h) = 0$ and $\theta(n, h) = \tilde{\theta}\}$ and $\{\langle n, h\rangle: \dot{\theta}^2(n, h) = 0$ and $\theta(n, h) = -\tilde{\theta}\}$ are finite.

In particular, (B) is satisfied if all functions of (3.1, H) are real analytic and $\tilde{\theta} < \theta(n_0, h_0)$.



THEOREM 3.3. Periodic solutions of the Hodgkin–Huxley equations with $m = m_\infty(V)$. *Assume Hypothesis* (3.1, CUBIC, H) *and* (3.2, PER, $\tilde{\theta}$, H). *Then* (3.1, H, $\tilde{\theta}$, $\epsilon$) *admits a periodic solution for all small* $\epsilon > 0$.

In the proof of 3.3 we verify hypothesis (2.3, PER, $\tilde{\theta}$) using Lemmas 3.4 and 3.5.

LEMMA 3.4. Analysis of $(\mathrm{H})_1$ and $(\mathrm{H})_2$. *Assume Hypothesis* (3.1, CUBIC, H).

$(\mathrm{H})_1$: (A) $\langle n_0, h_0\rangle$ *is the unique rest point of* $(\mathrm{H})_1$. *Both its eigenvalues are negative.*

(B) *There exist increasing* $C^1$ *functions* $h_a$, $h_b$ *such that in* $\mathrm{cl}(\pi_1)$:

$$n_\infty(V_1(n, h)) = n \quad \text{iff} \quad h = h_a(n),$$

*and*

$$h_\infty(V_1(n, h)) = h \quad \text{iff} \quad h = h_b(n).$$

*Moreover,* $\mathrm{dom}(h_a) \subseteq (0, 1)$; $h_a(n) = 0$ *for some* $n$; $\mathrm{range}(h_b) \subseteq (0, 1)$; *and* $1 \in \mathrm{dom}(h_b)$.

(C) $\langle n, h\rangle \in S(\langle n_0, h_0\rangle)$ *if* $\langle n, h\rangle \cdot^1 t$ *is defined for all* $t \geqslant 0$. *All but two solutions approach* $\langle n_0, h_0\rangle$ *in* $\{\dot{n}^1 > 0, \dot{h}^1 > 0\} \cup \{\dot{n}^1 < 0, \dot{h}^1 < 0\}$.

$(\mathrm{H})_2$. (D) $V_2(n, h) > 0$ *for* $\langle n, h\rangle \in \Pi_2$.

(E) $\Pi_2$ *contains no rest points of* $(\mathrm{H})_2$.

(F) *There exist increasing* $C^1$ *functions* $h_c$, $h_d$ *such that in* $\mathrm{cl}(\Pi_2)$:

$$n_\infty(V_2(n, h)) = n \quad \text{iff} \quad h = h_c(n);$$

*and*

$$h_\infty(V_2(n, h)) = h \quad \text{iff} \quad h = h_d(n).$$

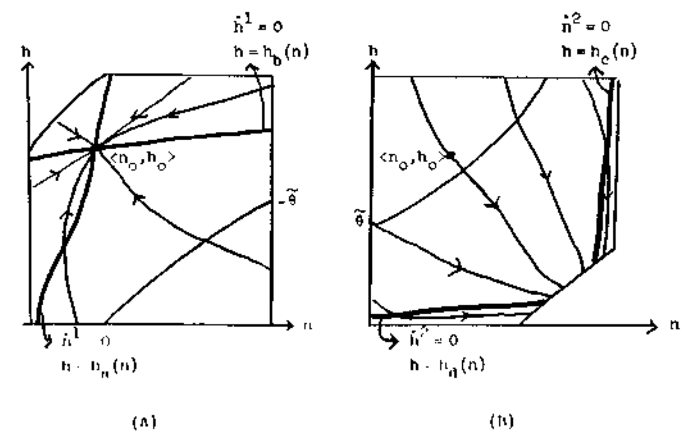


FIG. 13. (A) (3.2, H)$_1$ on $\Pi_1$. (B) (3.2, H)$_2$ on $\Pi_2$.

*In addition, there exist* $0 < n_d < n_c < n_c' < 1$, *with* $n_c > n_0$, *such that* $\mathrm{dom}(h_d) = [0, n_d]$; $\mathrm{dom}(h_c) = [n_c, n_c']$; $h_d(n_d) < h_0$; *and* $h_c(n_c') = 1$.

(G) *For every* $\langle n, h\rangle \in \Pi_2$ *there exists* $t > 0$ *such that*

$$\langle n, h\rangle \overset{2}{\cdot} t \in \{\langle n, h\rangle : \partial G/\partial V\ (V_2(n, h), n, h) = 0\}.$$

Lemma 3.5. Existence of $M$. *Let* $f : [0, 1] \to (0, 1)$ *be a piecewise-continuous, order-preserving function. Then there exists* $M$, *a closed subinterval of* $[0, 1]$, *such that* $f$ *is continuous on* $M$ *and* $f(M) \subsetneq \mathrm{int}(M)$.

## 4. Fast Variables: Periodic Solutions of the Hodgkin–Huxley Equations

In this section we consider the system:

$$\begin{aligned} \dot V &= W,\\ \dot W &= \theta W + g(V, y, z),\\ \dot y &= \epsilon\theta^{-1}h(V, y, z),\\ \dot z &= \delta^{-1}\theta^{-1}q(V, y, z), \end{aligned} \tag{4.1}$$

where $V \in [V_\alpha, V_\beta] \subseteq \mathbb{R}$; $y \in \Omega_y(\text{compact}) \subseteq \mathbb{R}^l$; $z \in \Omega_z(\text{compact}) \subseteq \mathbb{R}^m$; $\theta, \epsilon, \delta > 0$; and $g, h, q \in C^2$.

Hypothesis (4.1, FAST). (A) *There exists a* $C^1$ *function* $z(V, y)$: $[V_\alpha, V_\beta] \times$ *such that* $q(V, y, z(V, y)) = 0$.

(B) *For fixed* $\langle V, y\rangle \in [V_\alpha, V_\beta] \times \Omega_y$, *the* $m$ *eigenvalues of* $q(V, y, z)$ *(as a function of* $z$*) at* $z = z(V, y)$ *have negative real parts.*

Assume that (4.1) satisfies Hypothesis (4.1, FAST) and let (4.2) be the associated system:

$$\begin{aligned} \dot V &= W,\\ \dot W &= \theta W + G(V, y),\\ \dot y &= \epsilon\theta^{-1}H(V, y), \end{aligned} \tag{4.2}$$

where $G(V, y) \equiv g(V, y, z(v, y))$ and $H(V, y) \equiv h(V, y, z(v, y))$.

If $\delta$ is small, it is reasonable to expect that a bounded solution of (4.1) would stay close to the corresponding solution of (4.2).

This is in fact the case [1]; in particular, we have Theorem 4.2.

Theorem 4.2. Periodic solutions with fast variables.

*Assume that* (4.1) *satisfies Hypothesis* (4.1, FAST) *and that the associated system* (4.2) *satisfies Hypotheses* (2.1, CUBIC) *and* (2.3, PER, $\tilde\theta$). *Then* (4.1, $\tilde\theta$, $\epsilon$, $\delta$) *admits a periodic solution if* $\epsilon, \delta > 0$ *are small.*



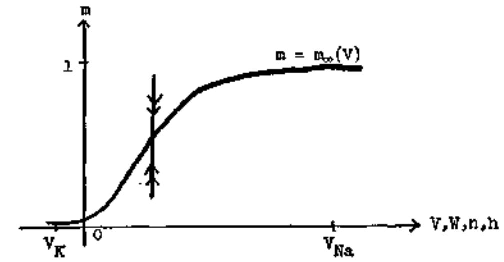


Fig. 14. The Hodgkin–Huxley equations with $\delta$ small.

Corollary 4.3. Periodic solutions of the Hodgkin–Huxley equations. *Let* (HH) *be the system:*

$$\begin{aligned} \dot V &= W,\\ \dot W &= \theta W + g(V, m, n, h),\\ \dot n &= \epsilon\theta^{-1}\gamma_n(V)(n_\infty(V) - n),\\ \dot h &= \epsilon\theta^{-1}\gamma_h(V)(h_\infty(V) - h),\\ \dot m &= \delta^{-1}\theta^{-1}\gamma_m(V)(m_\infty(V) - m). \end{aligned} \tag{4.3, HH}$$

*Assume that* $\gamma_m > 0$ *and* $0 < m_\infty < 1$ *and that the associated system* (3.1, H) $(m = m_\infty(V))$ *satisfies Hypotheses* (3.1, CUBIC, H) *and* (3.2, PER, $\tilde\theta$, H). *Then* (HH, $\tilde\theta$, $\epsilon$, $\delta$) *admits a periodic solution for all small* $\epsilon, \delta > 0$ *if* $0 < \tilde\theta < \theta(n_0, h_0)$; *and* (HH, $\tilde\theta$, $\epsilon$, $\delta$) *admits a periodic solution if* $\tilde\theta \geqslant \theta(n_0, h_0)$ *and* $F_1$ *is defined on* $F_2(\{\theta(n, h) = \tilde\theta\})$.

## 5. Proofs

Proofs of (1.2), (1.4), (2.2), and (3.4) appear in [1].

(2.4) We shall verify Hypothesis (1.3, PER) for the system (2.2, $\tilde\theta$, $\epsilon$) with $\epsilon$ small. Theorem 2.4 then follows from Theorem 1.4.

*Step* 1. $F_2 \mid M$ and $F_1 \mid F_2(M)$ are homeomorphisms.

*Proof.* (2.3, PER, $\tilde\theta$) (D) implies that no trajectory of $(\mathrm{H})_2$ crosses $\{\theta(y) = \tilde\theta\}$ more than once. Thus, $F_2$ is 1–1 and hence a homeomorphism. Similarly, $F_1$ is 1–1.

*Step* 2. Choose $L \subsetneq \mathrm{int}(M)$ such that $F_1 \circ F_2(M) \subsetneq \mathrm{int}(L)$ and $L$ is homeomorphic to $[0, 1]^{l-1}$. Then there exist blocks $A_i \subseteq \Pi_1 \cap \Pi_2$ for $(2.3)_i$ $(i = 1, 2)$ and $\tau > 0$ such that:

(i) No set $y \cdot^i [0, \infty)$ is contained in $A_i$ $(i = 1, 2)$.

(ii) $a_1^- \subseteq \{y: \theta(y) > \tilde{\theta}\}$ and $a_2^- \subseteq \{y: \theta(y) < -\tilde{\theta}\}$.

(iii) $A_1 \cap (F_1 \circ F_2(L) \cdot^1 [-\tau, \tau]) \subseteq \operatorname{int}(A_2)$; $A_2 \cap (F_2(L) \cdot^2 [-\tau, \tau]) \subseteq \operatorname{int}(A_1)$;

and each set is homeomorphic to $[0, 1]^l$.

Moreover,

$$F_1 \circ F_2(L) \cdot^1 (0, \tau] \subseteq \{\theta(y) > \tilde{\theta}\};$$
$$F_1 \circ F_2(L) \cdot^1 [-\tau, 0) \subseteq \{\tilde{\theta}/2 < \theta(y) < \tilde{\theta}\};$$
$$F_2(L) \cdot^2 (0, \tau] \subseteq \{\theta(y) < -\tilde{\theta}\};$$

and

$$F_2(L) \cdot^2 [-\tau, 0) \subseteq \{-\tilde{\theta}/2 > \theta(y) > -\tilde{\theta}\}.$$

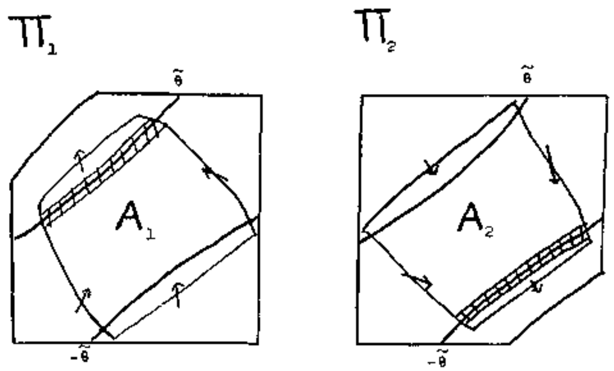


FIG. 15. $A_1$ and $A_2$ with $l = 2$, $M = \{\theta(y) = \tilde{\theta}\}$.

*Proof.* Fix $y \in \operatorname{int}(M)$. If $V$ is a neighborhood of $y$ in $\operatorname{int}(M)$, $F_2(V)$ is a neighborhood of $F_2(y)$ in $F_2(M)$. Thus there exists $T_y > 0$ such that $F_2(y) \cdot^2 (0, T_y] \cap \{\theta(y) = -\tilde{\theta}\} = \phi$ (Fig. 16).

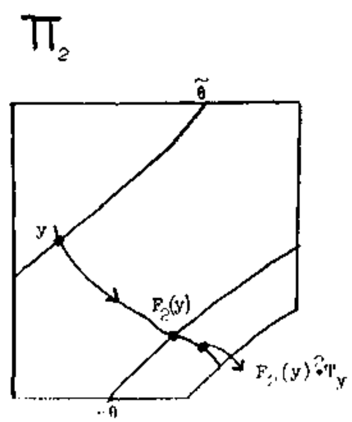


FIGURE 16



Let $U$ be a neighborhood of $F_2(y)$ in $F_2(M)$ such that $y \cdot^2 t \notin U - \{F_2(y)\}$ for any $t$. Choose $V$ such that $F_2(V) \subseteq U$. Then $C \equiv \{\tilde{y} \cdot^2 t: \tilde{y} \in V$ and $t \in [0, T_2(\tilde{y})]\}$ is a cylinder in $\Pi_2$, and $y \cdot^2 t \in C$ iff $t \in [0, T_2(y)]$. Thus $F_2(y) \cdot^2 (0, T_y] \subseteq \{\theta(y) < -\tilde{\theta}\}$. Clearly $F_2(y) \cdot^2 [-T_y, 0) \subseteq \{-\tilde{\theta}/2 > \theta(y) > -\tilde{\theta}\}$ if $T_y$ is small. Thus there exists $T > 0$ such that:

$$F_2(L) \cdot^2 (0, T] \subseteq \{\theta(y) < -\tilde{\theta}\}$$

and

$$F_2(L) \cdot^2 [-T, 0) \subseteq \{-\theta/2 > \theta(y) > -\tilde{\theta}\}.$$

Similarly, if $T$ is small,

$$F_1 \circ F_2(L) \cdot^1 (0, T] \subseteq \{\theta(y) > \tilde{\theta}\}$$

and

$$F_1 \circ F_2(L) \cdot^1 [-T, 0) \subseteq \{\tilde{\theta}/2 < \theta(y) < \tilde{\theta}\}.$$

We now construct $A_2$. If $C_2 \equiv \bigcup_{y \in L} y \cdot^2 [-T, T_2(y) + T]$, there exists a homeomorphism $h: [0, 1] \times [0, 1]^{l-1} \to C_2$ such that:

$$h(\{0\} \times [0, 1]^{l-1}) = L \overset{2}{\cdot} (-T),$$

and

$$h(\{1\} \times [0, 1]^{l-1}) = \bigcup_{y \in L} y \overset{2}{\cdot} (T_2(y) + T).$$

Choose $\sigma > 0$ such that $F_1 \circ F_2(L) \subseteq h([0, 1] \times (\sigma, 1 - \sigma)^{l-1})$, and let

$$A_2 \equiv \bigcup_{s \in [0,1]} h(s, [(1 - s)\sigma, 1 - (1 - s)\sigma]^{l-1}).$$

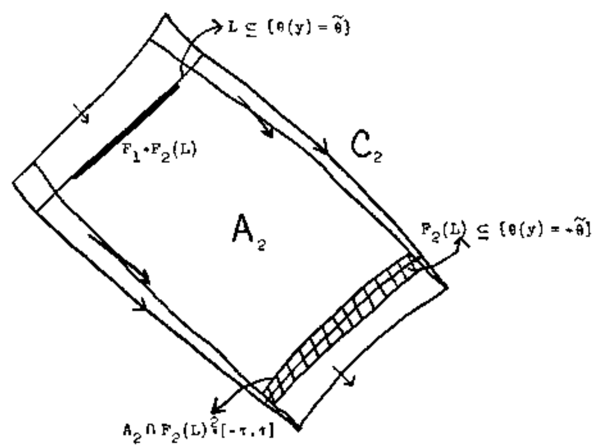


FIG. 17. $A_2$.

Then $A_2$ is a block for $(2.3)_2$; no set $y \cdot^2 [0, \infty) \subseteq A_2$;

$$a_2^- = \bigcup_{y \in L} y \overset{2}{\cdot} (T_2(y) + T) \subseteq \{\theta(y) < -\tilde{\theta}\};$$

and if $\tau \leqslant T$ is small, $A_2 \cap (F_2(L) \cdot^2 [-\tau, \tau])$ is homeomorphic to $[0, 1]^l$ and contained in $\text{int}(C_1)$, where

$$C_1 \equiv \bigcup_{y \in F_2(L)} y \overset{1}{\cdot} [-T, T_1(y) + T].$$

The construction of $A_1$ is similar with

$$A_2 \cap F_2(L) \overset{2}{\cdot} [-\tau, \tau]) \subseteq \text{int}(A_1);$$

$$\text{no set } y \overset{1}{\cdot} [0, \infty) \subseteq A_1;$$

$$a_1^- = \bigcup_{y \in F_2(L)} y \overset{1}{\cdot} (T_1(y) + T) \subseteq \{\theta(y) > \tilde{\theta}\};$$

and $A_1 \cap F_1 \circ F_2(L) \cdot^1 [-\tau, \tau]$ homeomorphic to $[0, 1]^l$ and contained in $\text{int}(A_2)$ (shrinking $\tau$ if necessary).

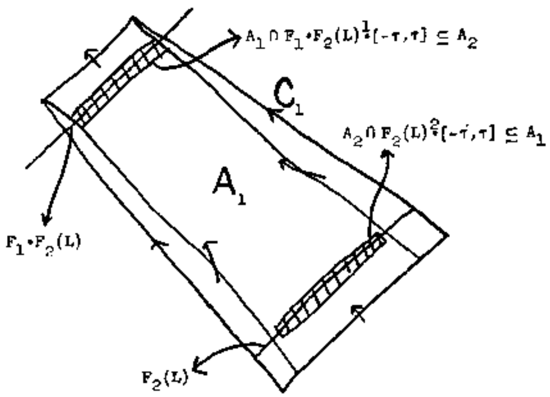


FIG. 18. $A_1$.

*Step* 3. Use $A_1$, $A_2$ to construct blocks $B_1$, $B_2$ for $(2.2, \tilde{\theta}, \epsilon)$ satisfying hypothesis (1.3, PER).

*Proof.* We shall first construct a family of blocks $\{\beta_s : s \in (0, 1]\}$ and let $B_1 \equiv \beta_s$ for some $s$, chosen after $B_2$ is constructed.



For fixed $y$, $(2.2, \tilde{\theta}, 0)$ defines a system:

$$\begin{aligned} \dot{V} &= W, \\ \dot{W} &= \tilde{\theta} W + G(V, y). \end{aligned} \qquad (*)_y$$

(Compare Example 1.1.)

Since $G(V_1(y), y) = 0$ and $\partial G / \partial V(V_1(y), y) > 0$, for all small $c > 0$ there exists a block $B^1_{c,y}$ for $(*)_y$ containing $\langle V_1(y), 0\rangle$ with the properties described in Example 1.1.

For $\bar{c} > 0$, choose a $C^1$ function $c(y, s)$: $\Pi_1 \cap \Pi_2 \times (0, 1] \to (0, \bar{c}]$ such that

$$\begin{aligned} c(y, s) &= \bar{c}, && \text{if } \theta(y) < 0 \\ &= s\bar{c}, && \text{if } \theta(y) > \tilde{\theta}/2. \end{aligned}$$

Then if $\bar{c}$, $\epsilon$ are small, $\beta_s \equiv \{\langle V, W, y\rangle : y \in A_1 \text{ and } \langle V, W\rangle \in B^1_{c(y,s),y}\}$ is a block for $(2.2, \tilde{\theta}, \epsilon)$ such that no set $\langle V, W, y\rangle \cdot [0, \infty) \subseteq \beta_s$ and $\beta_s^\pm = \beta_s \cap \{\langle V, W, y\rangle : u \in a^\pm \text{ or } \langle V, W\rangle \in b^{1\pm}_{c(y,s),y}\}$ ($B_1$ of Fig. 2). (Let $\phi_s^\pm$ refer to $\beta_s$.) $U(\langle V_2(y), 0, y\rangle)$ is carried into each $\beta_s$ if $y \in A_2 \cap [F_2(L) \cdot^2 [-\tau', \tau']]$ for some $\tau' \leqslant T$.

We now construct $B_2$. As before, for fixed $y \in \Pi_2$ and small $c > 0$ there exists a block $B^2_{c,y}$ for $(*)_y$ containing $\langle V_2(y), 0\rangle$. Let

$$B_2 \equiv \{\langle V, W, y\rangle : y \in A_2 \text{ and } \langle V, W\rangle \in B^2_{c,y}\}.$$

If $\tau$, $c$, $\epsilon$ are small, $B_2$ is a block for $(2.2, \tilde{\theta}, \epsilon)$ such that no positive half solution is contained in $B_2$; $\beta_1 \cap B_2 = \varnothing$; and $b_2^\pm = B_2 \cap \{\langle V, W, y\rangle : y \in a_1^\pm \text{ or } \langle V, W\rangle \in b^{2\pm}_{c,y}\}$. Moreover, if $\Delta \equiv \{\langle V, W, y\rangle \in b_2^- : W \leqslant 0 \text{ and } y \in F_2(L) \cdot^2 [-\tau', \tau']\}$, $\delta_0 \equiv \Delta \cap \{F_2(L) \cdot^2 \tau'\}$, and $\delta_1 \equiv \Delta \cap F_2(L) \cdot^2 (-\tau')$, then for every $s$

$$\phi_s^1(\text{cl}(\Delta)) \subseteq \text{int}(\beta_s^+);$$

$$\phi_s \circ \phi_s^1(\delta_0) \subseteq \{W > 0\};$$

and

$$\phi_s \circ \phi_s^1(\delta_1) \subseteq \{W < 0\}.$$

Finally, we may now choose $\tau < \tau'$ and $s$ such that if $B_1 \equiv \beta_s$; $\Gamma \equiv \{\langle V, W, y\rangle \in b_1^- : W \leqslant 0 \text{ and } y \in F_1 \circ F_2(L) \cdot^1 [-\tau, \tau]\}$; $\gamma_0 \equiv \Gamma \cap F_1 \circ F_2(L) \cdot^1 \tau$; and $\gamma_1 \equiv \Gamma \cap F_1 \circ F_2(L) \cdot^1 (-\tau)$, then

$$\phi_2(\text{cl}(\Delta)) \subseteq \text{int}(b_2^+),$$

$$\phi_2 \circ \phi_2^1(\gamma_0) \subseteq \{W > 0\},$$

and

$$\phi_2 \circ \phi_2^1(\gamma_1) \subseteq \{W < 0\}.$$

$B_1$ and $B_2$ now satisfy Hypothesis (1.3, PER).

(3.3) Lemma 3.4 implies that $\{\theta(n, h) = \tilde{\theta}\}$ is homeomorphic to $[0, 1]$ and $F_1 \circ F_2$ is a piecewise-continuous, order-preserving map from $\{\theta(n, h) = \tilde{\theta}\}$ into itself. Lemma 3.5 then implies that there exists $M \subseteq \{\theta(n, h) = \tilde{\theta}\}$ such that $M$ is homeomorphic to $[0, 1]$; $F_1 \circ F_2$ is continuous on $M$; and $F_1 \circ F_2(M) \subsetneq \mathrm{int}(M)$. Since $F_1 \circ F_2$ is order-preserving, $F_1$ and $F_2$ are each continuous. Thus Hypothesis (2.3, PER, $\tilde{\theta}$) is satisfied.

(3.5) Let $0 \leqslant x_1 < \cdots < x_N \leqslant 1$ be the points of discontinuity of $f$. If $N = 0$ (i.e., $f$ is continuous), the lemma holds with $M = [0, 1]$. Assume the lemma for $N \leqslant K$ and prove it for $N = K + 1$. Suppose that $f$ is continuous on $[0, x_1]$. (The proof for other cases is similar.) If $f(x_1) < x_1$, let $M = [0, x_1]$ (Fig. 20(A)). If $f(x_1) \geqslant x_1$, choose $\bar{x}$ such that $\bar{x}_1 < x < \inf\{f(x): x \in (x_1, 1]$ (Fig. 20(B)). Then $f \mid [\bar{x}, 1]$ has at most $K$ discontinuities and $f([\bar{x}, 1]) \subsetneq (\bar{x}, 1)$. The lemma follows from the inductive hypothesis.

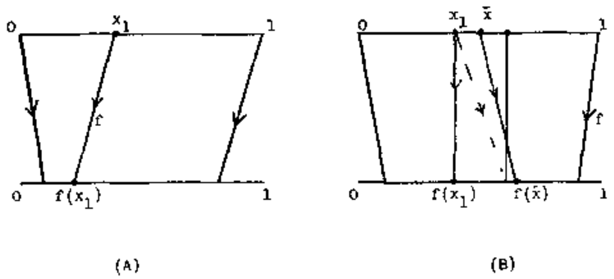


FIG. 19. (A) $f(x_1) < x_1$. (B) $f(x_1) \geqslant x_1$.

(4.2) The methods of [1] imply that whenever (4.2) satisfies (2.1, CUBIC) and (2.3, PER, $\tilde{\theta}$) then there exist blocks $B_1$, $B_2$ for (4.1) which satisfy Hypothesis (1.3, PER) if $\delta$ is small. The result then follows from Theorem 1.4.

(4.3) For fixed $V$, $-\gamma_m(V) < 0$ is the eigenvalue of $\gamma_m(V)(m_\infty(V) - m)$ when $m = m_\infty(V)$. Thus (HH) satisfies Hypothesis (4.1, FAST). The corollary follows from 4.2 and the proof of 3.3.

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