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A Prediction Theory for Some Nonlinear Functional-Differential Equations II. Learning of Patterns*

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INTRODUCTION

This paper studies the following system of nonlinear difference-differential equations:

$$\dot{x}_i(t) = -\alpha x_i(t) + \beta \sum_{m=1}^{n} x_m(t-\tau)\, y_{mi}(t) + I_i(t), \qquad (1)$$

$$y_{jk}(t) = z_{jk}(t)\left[\sum_{m=1}^{n} z_{jm}(t)\right]^{-1}, \qquad (2)(*)$$

and

$$\dot{z}_{jk}(t) = -u z_{jk}(t) + \beta x_j(t-\tau)\, x_k(t), \qquad (3)$$

where $i, j, k = 1, 2,..., n$, and $\beta > 0$. We will establish global limit and oscillation theorems for the nonnegative solutions of (*) when (*) has any fixed number of variables ($n \geq 2$) and τ is any fixed nonnegative time lag.

(*) arises as an example of a nonstationary prediction theory, or learning theory, whose goal is to discuss the prediction of individual events, in a fixed order, and at prescribed times ([1], [2]). In this theory, (*) describes a machine M subjected to inputs $C = (I_1, I_2, ..., I_n)$ by an experimenter E, who records the outputs $X = (x_1, x_2, ..., x_n)$ created thereby. E has only the inputs C and outputs X at his disposal with which to describe (*), and in terms of these variables (*) takes the form

$$\dot{X}(t) = -\alpha X(t) + B(X_t)\, X(t-\tau) + C(t), \qquad (4)$$

where $B(X_t)$ is a matrix of nonlinear functionals of $X(w)$ evaluated at all past times $w \in [-\tau, t]$ with entries

$$B_{ij}(t) = \frac{z_{ji}(0) + \beta \int_0^t e^{uv} x_j(v-\tau)\, x_i(v)\, dv}{\sum_{m=1}^{n} \left[z_{jm}(0) + \beta \int_0^t e^{uv} x_j(v-\tau)\, x_m(v)\, dv\right]}. \qquad (5)$$

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The machine M therefore obeys the functional-differential equations (4)-(5), and $B(X_t)$ contains the "memory" of M. Our global limit and oscillation theorems for (*) can be interpreted as learning experiments performed by E on M to study how M learns, remembers what it has learned, and reacts to test inputs in recall experiments. In particular, (*) can learn a spatial pattern in "black and white" of arbitrary size and complexity (see [3]).

The prediction theory in [1] introduces infinitely many nonlinear systems. Each system is characterized by an $n \times n$ "coefficient" matrix $P = \| p_{ij} \|$ which is *semistochastic*; that is, $p_{ij} \geq 0$ and $\sum_{m=1}^{n} p_{im} = 0$ or 1. (*) is characterized by the stochastic matrix with entries $p_{ij} = (1/n)$. This matrix can be realized as a probabilistic network G [4], and (*) can be interpreted as a cross-correlated flow over G [5] in the following way.

G consists of n vertices $V = \{v_i : i = 1, 2,..., n\}$ and n^2 directed edges $E = \{e_{jk} : j, k = 1, 2,..., n\}$, where e_{jk} has v_j as its initial vertex and v_k as its terminal vertex. The coefficient matrix P assigns the weight $p_{jk} = (1/n)$ to e_{jk}. Since every vertex v_i is connected to every other vertex v_j with equal weight, the graph G is *complete*. Since v_i is also connected to itself, G is a complete graph *with loops*. We illustrate this graph in the case $n = 3$ in Fig. 1.

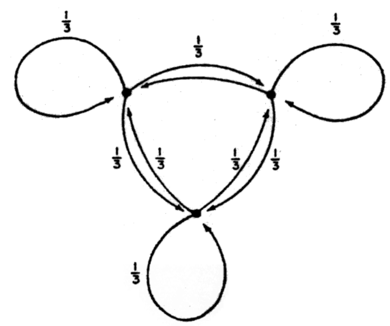

FIG. 1

We describe (*) as a flow over this complete graph with loops in the following way. At every time t, $x_m(t)$ is the state of a process going on at vertex v_m, and $y_{mi}(t)$ is the state of a process going on at the arrowhead of e_{mi}, $i, m = 1, 2,..., n$. At every time $w = t - \tau$, a quantity $\beta x_m(w)$ flows, or is "transmitted," from v_m along e_{mi} at a finite velocity and reaches the arrowhead of e_{mi} at time $w + \tau = t$. This quantity instantaneously activates the $y_{mi}(t)$ process in the arrowhead, and a total magnitude

$$\beta x_m(t-\tau)\, y_{mi}(t) \qquad (6)$$

is released from the arrowhead and reaches v_i at time t. This is true for every $m = 1, 2$ n. The total input to v_i from all vertices v_m at time t is the sum

$$\beta \sum_{m=1}^{n} x_m(t - \tau)\, y_{mi}(t) \tag{7}$$

of the inputs (6). By (1), $x_i(t)$ changes at a rate equal to the sum of this total input, of a "spontaneous decay" term $-\alpha x_i(t)$, and of the input $I_i(t)$ controlled by E.

Since $\sum_{i=1}^{n} y_{mi}(t) = 1$ whenever the initial data of (*) is positive [1], the total output from v_m reaching all vertices v_i at time t is simply

$$\beta \sum_{i=1}^{n} x_m(t - \tau)\, y_{mi}(t) = \beta x_m(t - \tau). \tag{8}$$

We call the flow which (*) describes a "cross-correlated" flow because of the following interpretation of the functions $z_{jk}(t)$ in (3). At every time t, the quantity $\beta x_j(t - \tau)$ reaches the arrowhead of e_{jk} from v_j. Also the arrowhead of e_{jk} impinges on v_k, whose process has the magnitude $x_k(t)$ at time t. $z_{jk}(t)$ "cross-correlates" the two quantities $\beta x_j(t - \tau)$ and $x_k(t)$ impinging on the arrowhead by changing at a rate equal to $\beta x_j(t - \tau)\, x_k(t)$ minus a spontaneous decay term $-u z_{jk}(t)$.

The term $y_{jk}(t)$ which actually controls the size of the input $\beta x_j(t - \tau)\, y_{jk}(t)$ from v_j to v_k is formed from $z_{jk}(t)$ normalized by the sum of all $z_{jm}(t)$ corresponding to edges e_{jm} leading away from v_j, $m = 1, 2,..., n$, as in (2). This normalization of cross-correlating functions has a profound effect on the behavior of (*) that is due, for example, to the fact that the *total* output from v_m is independent of all cross-correlating functions, as (8) shows.

2. A Probabilistic Equation

Our main results concerning (*) describe the global limiting and oscillatory behavior of the ratios

$$y_{jk}(t) \quad z_{jk}(t) \left[\sum_{m=1}^{n} z_{jm}(t)\right]$$

and the correspondingly defined ratios

$$X_k(t) = x_k(t) \left[\sum_{m=1}^{n} x_m(t)\right]$$

as $t \to \infty$ when is any nonnegative time lag and n is any positive integer,

which we take greater than 1 to avoid trivialities. The special case $\tau = 0$ and $n = 2$ is studied in [2]. We will investigate these ratios only when the initial data of (*) is continuous and *nonnegative* in $[-\tau, 0]$, since only such initial data has a prediction theoretical interpretation [1]. It is then readily shown [1] that the solution of (*) exists and is unique, continuously differentiable, and nonnegative in $(0, \infty)$. We also suppose for convenience that $\sum_{m=1}^{n} x_m(v) > 0, v \in [-\tau, 0]$, and that $z_{jk}(0) > 0, j, k = 1, 2,..., n$. Then $\sum_{m=1}^{n} x_m(t) > 0$ and $z_{jk}(t) > 0$ for all $t \geq 0$. The sets $\{y_{jm}(t) : m = 1, 2,..., n\}$ and $\{X_m(t) : m = 1, 2,..., n\}$ of ratios therefore form probability distributions for every $t \geq 0$.

We will find conditions under which desirable limiting properties of these probabilities become easier to guarantee as τ *increases*. Moreover, several of these probabilities oscillate no more than once as $t \to \infty$ no matter how large τ is taken. These results fall into two general cases corresponding to special choices of the inputs I_j. In the first case, no inputs whatsoever perturb (*); that is, (*) is *input-free*. In the second case, inputs of the special form $I_j(t) = \theta_j I(t)$, where $\{\theta_m : m = 1, 2,..., n\}$ is an arbitrary probability distribution, do perturb (*) and continue to do so at arbitrarily large times. These cases can be treated because (*) can be transformed into a more tractable system of integro-difference-differential equations for the probabilities $X_i(t)$ and $y_{jk}(t)$ themselves. In this new system, the sums $I = \sum_{m=1}^{n} I_m$ and $x = \sum_{m=1}^{n} x_m$ play a significant role.

Proposition 1. The probabilities X_i and y_{jk} obey the following equations.

$$\dot{X}_i(t) = A(t) \sum_{m=1}^{n} X_m(t - \tau)[y_{mi}(t) \quad X_i(t)] + B(t)[\theta_i(t) - X_i(t)], \tag{9}$$

and

$$\dot{y}_{jk}(t) = C_j(t)[X_k(t) - y_{jk}(t)], \tag{10}$$

where

$$A(t) = \frac{\beta x(t - \tau)}{x(t)} \tag{11}$$

$$B(t) = \frac{I(t)}{x(t)}, \tag{12}$$

$$\theta_i(t) = \frac{I_i(t)}{I(t)}. \tag{13}$$

and

$$C_j(t) \quad \frac{d}{dt} \log \left[\frac{1}{\beta} \sum_{m=1}^{n} z_{jm}(0) + \int X_j(v - \tau)\, e^{uv} x(v - \tau)\, x(v)\, dv\right] \tag{14}$$

PROOF To derive (9), differentiate $X_i = (x_i/x)$. Then

$$\dot{X}_i = \frac{1}{x}\left(\dot{x}_i - x_i \frac{\dot{x}}{x}\right).$$

To evaluate this equation, note that summing over $i = 1, 2, \ldots, n$ in (1) and invoking positivity shows that x is a positive solution of the *linear* equation

$$\dot{x}(t) = -\alpha x(t) + \beta x(t - \tau) + I(t). \tag{15}$$

When (1) and (15) are applied, we find

$$\dot{X}_i = \frac{1}{x}\left[-\alpha x_i + \beta \sum_{m=1}^{n} x_m(t - \tau) y_{mi} + I_i - x_i\left(-\alpha + \frac{\beta x(t - \tau) + I}{x}\right)\right]$$

$$= \frac{\beta}{x}\left[\left(\sum_{m=1}^{n} x_m(t - \tau) y_{mi} - \frac{x_i x(t - \tau)}{x}\right) + \left(I_i - \frac{x_i I}{x}\right)\right]$$

$$= \frac{\beta x(t - \tau)}{x}\left(\sum_{m=1}^{n} X_m(t - \tau) y_{mi} - X_i\right) + \frac{I}{x}(\theta_i - X_i)$$

$$= A \sum_{m=1}^{n} X_m(t - \tau)[y_{mi} - X_i] + B(\theta_i - X_i),$$

which proves (9). To prove (10), differentiate $y_{jk} = [z_{jk}/z^{(j)}]$, where $z^{(j)} = \sum_{m=1}^{n} z_{jm}$. Then

$$\dot{y}_{jk} = \frac{1}{z^{(j)}}\left[\dot{z}_{jk} - z_{jk}\frac{\dot{z}^{(j)}}{z^{(j)}}\right] \tag{16}$$

To evaluate $(\dot{z}^{(j)}/z^{(j)})$ in (16), sum over $k = 1, 2, \ldots, n$ in (3) to find that

$$\dot{z}^{(j)} = -uz^{(j)} + \beta x_j(t - \tau)\, x. \tag{17}$$

Substitute (3) and (17) into (16). Then

$$\dot{y}_{jk} = \frac{1}{z^{(j)}}\left[-uz_{jk} + \beta x_j(t - \tau)\, x_k - z_{jk}\left(-u + \frac{\beta x_j(t - \tau)\, x}{z^{(j)}}\right)\right]$$

$$= \frac{\beta x_j(t - \tau)}{z^{(j)}}\left[x_k - \frac{z_{jk} x}{z^{(j)}}\right]$$

$$= \frac{\beta x_j(t - \tau)\, x}{z^{(j)}}[X_k - y_{jk}]$$

$$= \frac{\beta X_j(t - \tau)\, x(t - \tau)\, x}{z^{(j)}}[X_k - y_{jk}].$$

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$$C_j(t) = \frac{\beta X_j(t - \tau)\, x(t - \tau)\, x}{z^{(j)}(t)}$$

it remains only to show that C_j can be written as in (14). This is easily seen when $z^{(j)}$ is written in integral form as

$$z^{(j)}(t) = e^{-ut}\left[z^{(j)}(0) + \beta \int_0^t X_j(v - \tau)\, e^{uv} x(v - \tau)\, x(v)\, dv\right].$$

REMARK. The only unknowns in (9)–(13) are X_i and y_{jk}, since x obeys (15) whose solution depends only on the known input I and initial data.

3. THE INPUT-FREE GRAPH

In this section, we study (9) and (10) when all inputs I_j are identically zero. Then (9) becomes

$$\dot{X}_i = A \sum_{m=1}^{n} X_m(t - \tau)[y_{mi} - X_i]. \tag{18}$$

Our main result concerning (10) and (18) discusses the limits and oscillations of X_i relative to the functions

$$y_i = \min\{y_{mi} : m = 1, 2, \ldots, n\}$$

and

$$Y_i = \max\{y_{mi} : m = 1, 2, \ldots, n\}.$$

The limiting behavior of (10) and (18) depends on the time lag τ only through the constant $\sigma(\tau) \equiv u + 2s(\tau)$, where $s(\tau)$ is the largest real part of the zeros of the characteristic exponential polynomial of

$$\dot{x}(t) = -\alpha x(t) + \beta x(t - \tau), \tag{19}$$

which is

$$R_\tau(s) = s + \alpha - \beta e^{-\tau s}.$$

THEOREM 1. *For any $n \geq 2$ and any $\tau \geq 0$ with $\sigma(\tau) > 0$, let* (10) *and* (18) *have arbitrary nonnegative and continuous initial data. Then*

(1) *(limiting behavior) the limits*

$$Q_i = \lim_{t \to \infty} X_i(t) \quad \text{and} \quad P_{jk} = \lim_{t \to \infty} y_{jk}(t)$$

exist and satisfy the equations

$$P_{ji} \quad Q_i, \qquad i, j = 1, 2, \ldots, n.$$

Moreover, $Q_i \in [m_i, M_i]$, *where*

$$m_i = \min\{X_i(0), y_i(0)\} \qquad \text{and} \qquad M_i = \max\{X_i(0), Y_i(0)\}$$

(2) (*oscillatory behavior*) *the functions* $\dot{y}_i$, $\dot{Y}_i$, $X_i - y_i$, *and* $X_i - Y_i$ *change sign at most once and not at all if* $y_i(0) \leqslant X_i(0) \leqslant Y_i(0)$. (*Derivatives of* y_i *or* Y_i *at times when two or more* y_{ki} *intersect are defined as right or left-handed derivatives in any systematic way.*)

We will prove the theorem in a series of lemmas. We begin with a lemma concerning the oscillations described in (2), since these do not depend on the sign of $\sigma(\tau)$. Then we use (2) along with some facts about the sum $x = \sum_{m=1}^{n} x_m$ to establish the assertions in (1) concerning limits when $\sigma(\tau) > 0$.

LEMMA 1. *For arbitrary nonnegative and continuous initial data, and any* $\sigma(\tau)$, *the functions* $\dot{y}_i$, $\dot{Y}_i$, $X_i - y_i$, *and* $X_i - Y_i$ *change sign at most once, and not at all if* $y_i(0) \leqslant X_i(0) \leqslant Y_i(0)$. *Moreover* $X_i(t)$, $y_i(t)$, *and* $Y_i(t)$ *lie in* $[m_i, M_i]$ *for all* $t \geqslant 0$.

PROOF. The following facts are obvious by an inspection of (10) and (18) using the positivity of A, X_j, and C_j, $j = 1, 2, \ldots, n$.

CASE 1. If, for any t_0, $X_i(t_0) \in [y_i(t_0), Y_i(t_0)]$, then $X_i(t) \in [y_i(t), Y_i(t)]$ for all $t \geqslant t_0$, where $y_i(t)$ is monotone increasing and $Y_i(t)$ is monotone decreasing for all $t \geqslant t_0$.

CASE 2. If $X_i(0) > Y_i(0)$, then $X_i(t)$ is monotone decreasing and all $y_{ki}(t)$ are monotone increasing until the first time $t = t_1 > 0$ at which $X_i(t) = Y_i(t)$. Thereafter $Y_i(t)$ is monotone decreasing and $y_i(t)$ is monotone increasing by Case 1, so that $\dot{Y}_i(t)$ changes sign at most once and $y_i(t)$ is always monotone increasing.

CASE 3. If $X_i(0) < y_i(0)$, then $X_i(t)$ is monotone increasing and all $y_{ki}(t)$ are monotone decreasing until the first time $t = t_1 > 0$ at which $X_i(t) = y_i(t)$. Thereafter $y_i(t)$ is monotone increasing by Case 1, so that $\dot{y}_i(t)$ changes sign at most once, and $Y_i(t)$ is always monotone decreasing. These alternatives are illustrated in Fig. 2.

Since Cases (1), (2), and (3) exhaust all possibilities, the assertions of the lemma are now evident.

The remainder of the proof requires estimates of the positive coefficients $A(t)$ and $C_j(t)$ in (10) and (18), as well as of $\dot{A}(t)$ and $\dot{C}_j(t)$, as $t \to \infty$. In providing these estimates, we always assume for convenience that all $z_{jk}(0)$ are positive and that $\sum_{m=1}^{n} x_m(v)$ is positive for all $v \in [-\tau, 0]$. The remaining

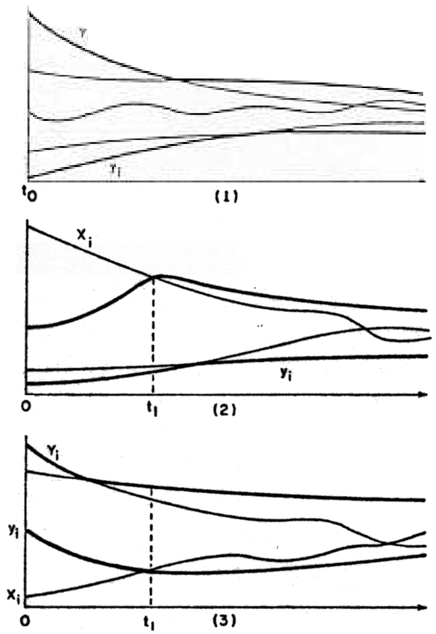

FIG. 2

cases with nonnegative initial data are easily treated. The basic fact from which these estimates arise is that the sum $x = \sum_{m=1}^{n} x_m$ obeys the *linear* difference-differential equation

$$\dot{x}(t) \quad -\alpha x(t) + \beta x(t \quad \tau), \tag{19}$$

which is independent of the probabilities y_{jk}. (19) is proved by simply summing over $i = 1, 2, \ldots, n$ in (1) and invoking positivity of all $z_{jk}(t)$ for $t \geqslant 0$.

Equation (19) has been thoroughly studied [6]. In the present account, we merely list the known facts we will need concerning (19) and derive some straightforward consequences from them. We will always work with the cases $\tau > 0$. The case $\tau = 0$ is obvious. Our first lemma concerns itself with the zeros of the characteristic exponential polynomial $R_\tau(s) = s + \alpha - \beta e^{-\tau s}$ of (19).

LEMMA 2. *For any fixed* $\tau > 0$, *the zero* $s_1(\tau)$ *of largest real part of* $R_\tau(s) = s + \alpha - \beta e^{-\tau s}$ *is real; that is,* $s_1(\tau) = s(\tau)$. *Moreover only finitely many zeros have a nonnegative real part.*

PROOF. Let $s = x + iy$ be a zero of $R_\tau(s)$. Then

$$x + \alpha - \beta e^{-\tau x} \cos \tau y = 0 \tag{20}$$

and

$$y + \beta e^{-\tau x} \sin \tau y = 0. \tag{21}$$

Write (20) in the form $y(x) = z_\theta(x)$, where $y(x) = x$, $z_\theta(x) = -\alpha + \beta\theta e^{-\tau x}$, and $\theta = \cos \tau y \in [-1, 1]$. For each fixed $\theta \in [-1, 1]$, we consider the graphs of $y(x)$ and $z_\theta(x)$ as functions of x. Every root of (20) must lie at the intersection of these graphs for some $\theta \in [-1, 1]$. For example, if $\alpha > \beta > 0$ we find Fig. 3.

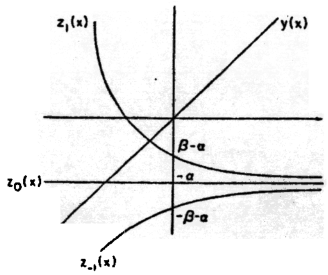

FIG. 3

It is clear from this diagram that the root x_1 of $y(x) = z_1(x)$ is a simple root and is the root with largest real part among all roots of the equations $y(x) = z_\theta(x)$, $\theta \in [-1, 1]$. When $\theta = 1$, $\cos \tau y = 1$ and $\sin \tau y = 0$. Thus by (21), the imaginary part y_1 corresponding to x_1 is $y_1 = -\beta e^{-\tau x} \sin \tau y_1 = 0$. The zero of largest real part of $R_\tau(s)$ is therefore a real and simple zero of $R_\tau(s)$.

Since $R_\tau(s)$ is a nontrivial entire function of s, only finitely many zeros of $R_\tau(s)$ can occur in any finite region of the s plane. For any zero $s_k = x_k + iy_k$ with nonnegative real part x_k, we have by (21) the inequality $| y_k | = \beta e^{-\tau x_k} | \sin \tau y_k | < \beta$, which along with Fig. 3 shows that at most finitely many zeros of $R_\tau(s)$ have a nonnegative real part.

The next Lemma describes a representation for solutions of (19).

LEMMA 3. *Let* x *be a solution of* (19) *with positive and continuously differentiable initial data in* $[0, \tau]$. *Then* x *can be written in the form*

$$x(t) = e^{s(\tau)t}[c_1 + e^{-kt}H(t)], \tag{22}$$

where k *and* c_1 *are positive and* H *is bounded.*

REMARK. If the initial data of (19) is merely continuous in $[-\tau, 0]$, then the solution of (19) is continuously differentiable in $(0, \tau]$, so that differentiability is not a restrictive assumption.

PROOF. The proof depends on the following standard representation theorem for the solutions of (19) ([6], p. 109).

Let $e^{s_k t}p_k(t)$ denote the residue of the function $e^{ts}p(s)[R_\tau(s)]^{-1}$ at a zero s_k of $R_\tau(s)$, where

$$p(s) = x(\tau)\, e^{-\tau s} + (s + \alpha) \int^{\tau} x(v)\, e^{-sv}\, dv.$$

Let $\{s_k\}$ be the sequence of zeros of $R_\tau(s)$ arranged in order of decreasing real parts. Then the solution x of (19) can be written in the infinite series

$$x(t) = \sum_{k=1}^{\infty} p_k(t)\, e^{s_k t} \qquad \text{for} \quad t > \tau.$$

This series converges uniformly for t in any finite interval $[t_0, t_0']$ where $t_0 > \tau$. Moreover if $\operatorname{Re} s_k \leq c < 0$ for all $k = 1, 2, \ldots$, then the series converges uniformly for $t \in [t_0, \infty)$, where $t_0 > \tau$.

All the zeros of $R_\tau(s)$ are simple zeros, since a nonsimple zero arises only when $1 = -\beta\tau \exp(1 + \alpha\tau) < 0$. In this case, the residue of $e^{ts}p(s)[R_\tau(s)]^{-1}$ at s_k is $e^{ts_k}p(s_k)[R_\tau'(s_k)]^{-1}$ and so

$$x(t) = \sum_{k=1}^{\infty} c_k e^{s_k t}, \tag{23}$$

where $c_k = p(s_k)[R_\tau'(s_k)]^{-1}$. c_k can be written in a simplified form as follows.

Since

$$R_\tau(s_k) = s_k + \alpha - \beta e^{-\tau s_k} = 0,$$

$p(s_k)$ can be written in the form

$$p(s_k) = e^{-\tau s_k}\left[x(\tau) + \beta \int_0^{\tau} x(v)\, e^{-v s_k}\, dv\right]$$

Noting also that

$$R_\tau'(s) = \quad + \beta\tau e^{-\tau s}$$

we find

$$c_k = \frac{e^{-\tau s_k}[x(\tau) + \beta \int_0^{\tau} x(v)\, e^{-v s_k}\, dv]}{1 + \beta\tau e^{-\tau s_k}}$$

and in particular, by Lemma

$$c_1 = \frac{e^{-\tau s(\tau)}[x(\tau) + \beta \int_0^\tau x(v)\, e^{-vs(\tau)}\, dv]}{1 + \beta\tau e^{-\tau s(\tau)}} \tag{24}$$

which is positive since the initial data of x is positive.

Also by Lemma 1, we know that only finitely many zeros of $R_\tau(s)$ have nonnegative real parts. (23) can therefore be written in the form

$$x(t) = e^{s(\tau)t}[c_1 + F(t) + G(t)] \tag{25}$$

where $e^{s(\tau)t}F(t)$ is the finite sum $\sum_{k=2}^m c_k e^{s_k t}$ over the terms $c_k e^{s_k t}$, $k \geqslant 2$, with $\mathrm{Re}\, s_k \geqslant 0$, and $e^{s(\tau)t}G(t)$ is the infinite sum $\sum_{k=m+1}^\infty c_k e^{s_k t}$ over the terms $c_k e^{s_k t}$ with $\mathrm{Re}\, s_k < 0$. Since $s(\tau) > \mathrm{Re}\, s_k$, $k > 1$, each of the summands in $F(t) = \sum_{k=2}^m c_k e^{(s_k - s(\tau))t}$ and in $G(t) = \sum_{k=m+1}^\infty c_k e^{(s_k - s(\tau))t}$ has a negative real part. We will use this fact to write (25) in the form

$$x(t) = e^{s(\tau)t}[c_1 + e^{-kt}H(t)], \tag{22}$$

where $k > 0$ and H is bounded.

We prove (22) by writing F and G separately as a product of an exponentially decreasing term and a bounded function and then adding. Thus we write $F(t) = e^{(x_2 - s(\tau))t}I(t)$, where $I(t) = \sum_{k=2}^m c_k e^{(s_k - x_2)t}$ is obviously bounded. In a similar fashion, we write $G(t)$ as $G(t) = e^{(x_2 - s(\tau))t}J(t)$, where $J(t) = \sum_{k=m+1}^\infty c_k e^{(s_k - x_2)t}$. It remains only to show that J is bounded. This fact is an immediate consequence of the following asymptotic formula for the zeros of $R_\tau(s)$ ([6], p. 416):

$$x = \frac{1}{\tau} \log \frac{\beta\tau}{2k\pi} + o(1)$$

and

$$y = \frac{\pi}{\tau}(2k - \tfrac{1}{2}) + o(1),$$

where k is any large integer. e^{xt} therefore has the asymptotic form

$$e^{xt} = \left[e^{o(1)} \frac{\beta\tau}{2\pi k}\right]^{t/\tau}$$

For sufficiently large t, $J(t)$ can therefore be shown to be bounded by comparison with the series $\sum_k (1/k^2)$. This completes the proof of Lemma 3.

We are now ready to estimate A, $\dot{A}$, C_j, and $\dot{C}_j$ as $t \to \infty$.

LEMMA 4 $\lim_{t\to\infty} A(t) = \beta e^{-\tau s(\tau)}$ *and* $\lim_{t\to\infty} \dot{A}(t) = 0$

PROOF. The first limit is obvious by (11) and (22). The second limit readily follows from the first because

$$\dot{A}(t) = \frac{\beta x(t-\tau)}{x(t)} \left[\frac{\dot{x}(t-\tau)}{x(t-\tau)} - \frac{\dot{x}(t)}{x(t)}\right],$$

which becomes by (19)

$$\dot{A}(t) = \frac{\beta^2 x(t-\tau)}{x(t)} \left[\frac{x(t-2\tau)}{x(t-\tau)} \quad \frac{x(t-\tau)}{x(t)}\right].$$

LEMMA 5. C_j *is bounded from above and below by positive constants* $\dot{C}_j$ | *is bounded.*

PROOF. The first assertion depends on Lemmas 1 and 3. By (14)

$$C_j(t) \quad \frac{X_j(t-\tau)\, e^{ut} x(t-\tau)\, x(t)}{d_j + \int_0^t X_j(v-\tau)\, e^{uv} x(v-\tau)\, x(v)\, dv},$$

where

$$d_j = \frac{1}{\beta} \sum_{m=1}^n z_{jm}(0) > 0$$

By Lemma 1, $X_j \geqslant m_j > 0$. Moreover, $X_j \leqslant$ Thus

$$C_j(t) \leqslant \frac{e^{ut} x(t-\tau)\, x(t)}{d_j + m_j \int_0^t e^{uv} x(v-\tau)\, x(v)\, dv}$$

By Lemma 3,

$$x(t-\tau)\, x(t) = e^{2s(\tau)t}[c_1{}^2 e^{-\tau s(\tau)} + e^{-kt}K(t)]$$

where $K(t)$ is nonnegative and bounded, so that

$$x(t-\tau)\, x(t) \leqslant e^{2s(\tau)t}[c_1{}^2 e^{-\tau s(\tau)} + K],$$

where $K = \sup\{K(t) : t \geqslant 0\}$. Now readily follows the inequality

$$C_j(t) \leqslant \frac{e^{\sigma(\tau)t}[c_1{}^2 e^{-\tau s(\tau)} + K]}{d_j + \frac{m_j c_1{}^2 e^{-\tau s(\tau)}}{\sigma(\tau)}(e^{\sigma(\tau)t} - 1)},$$

from which it is clear that $C_j(t)$ is bounded from above.

The second assertion follows from similar estimates. Since m $X_j \leqslant$

$$C_j(t) \geqslant \frac{m_j e^{ut} x(t-\tau)\, x(t)}{d_j + \int_0^t e^{uv} x(v-\tau)\, x(v)\, dv}$$

Thus

$$C_i(t) \quad \theta_i(t) \quad \frac{c_1^2 m_j e^{-\tau s(\tau)} e^{\sigma(\tau)t}}{d_i + \left(\frac{c_1^2 e^{-\tau s(\tau)} + K}{\sigma(\tau)}\right)(e^{\sigma(\tau)t} - 1)}$$

Since $\sigma(\tau) > 0$, letting

$$\theta \quad \frac{c_1^2 \sigma(\tau)\, e^{-\tau s(\tau)}}{2(c_1^2 e^{-\tau s(\tau)} + K)} \min\{m_j : j = 1, 2,..., n\}$$

completes the second assertion, with $C_i(t) \geqslant \theta > 0$ for large t.

The third assertion is proved in just the same way.

We will need one more lemma before studying the limiting behavior of (10) and (18). This result is an elementary fact about real-valued functions, which we prove for the sake of completeness.

Lemma 6. *Suppose $f(t) \to \lambda < \infty$ as $t \to \infty$ and $\ddot{f}$ is bounded. Then $\dot{f}(t) \to 0$ as $t \to \infty$.*

Proof. Suppose not. Then for some $\epsilon > 0$, there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $|\dot{f}(t_n)| \geqslant \epsilon$ for all n. We can suppose $\dot{f}(t_n) \geqslant \epsilon$ for all n without loss of generality. Since $\ddot{f}$ is bounded, these exists a δ such that $\dot{f} > (\epsilon/2)$ on infinitely many nonoverlapping intervals $I_n = [U_n, U_n + \delta]$ of length δ, where $\lim_{n\to\infty} U_n = \infty$. Thus

$$f(U_n + \delta) - f(U_n) \geqslant \frac{\epsilon\delta}{2}$$

for all n, and $f \nrightarrow \lambda < \infty$, which is a contradiction.

We can now complete the proof of Theorem 1 using Lemmas 1-6

4 Proof of Theorem

By Lemma the following three cases exhaust all possibilities.

Case 1. $X_i \geq Y_i$ for $t \geqslant 0$. Then by Lemma 1, X_i is monotone decreasing and all y_{ki} are monotone increasing. Hence all limits Q_i and P_{ki} exist and $Q_i \geq P_{ki}$. It is also readily shown using Lemmas 4 and 5 that $\ddot{X}_i$ is bounded, so that by Lemma 6, $\lim_{t\to\infty} \dot{X}_i(t) = 0$. Letting $t \to \infty$ in (18) and invoking Lemma 4, we now find

$$\sum_{m=1}^{n} Q_m(P_m \quad Q_i) \quad 0.$$

Since $Q_i \geq P_{ki}$ for all $k = 1, 2,..., n$, either $Q_k = 0$ or $P_{ki} = Q_i$ for all $k = 1, 2,..., n$. Since $Q_k \geq m_k > 0$ by Lemma 1, $P_{ki} = Q_i$ for all $k = 1, 2,..., n$.

Case 2. $X_i < y_i$ for all $t \geqslant 0$. The proof is the same as for Case with all inequalities reversed.

Case 3. $X_i(t) \in [y_i(t), Y_i(t)]$ for all $t \geq t_0$. By Lemma 1, Y_i decreases monotonically and y_i increases monotonically for $t \geq t_0$. Thus the limits $Y_i(\infty) = \lim_{t\to\infty} Y_i(t)$ and $y_i(\infty) = \lim_{t\to\infty} y_i(t)$ exist. If $Y_i(\infty) = y_i(\infty)$, we are done since then all Q_i and P_{ki} exist and equal $y_i(\infty)$. The only remaining case is $\epsilon_i \equiv Y_i(\infty) - y_i(\infty) > 0$. We now show that this case cannot arise.

Consider $y_i(t)$. We write $y_i(t)$ as $y_{k(t),i}(t)$ to explicitly display the index $k = k(t)$ of that $y_{ki}(t)$ which equals $y_i(t)$ at every t. We know that $\lim_{t\to\infty} y_{k(t),i}(t)$ exists and wish to conclude that $\lim_{t\to\infty} \dot{y}_{k(t),i}(t) = 0$ by Lemma 6. By Lemma 5, each $\dot{y}_{ki}(t)$ is bounded, and so the boundedness of $\ddot{y}_{k(t),i}(t)$ follows from the boundedness of all the $\ddot{y}_{ki}(t)$, which in turn is also a consequence of Lemma 5.

Since $\lim \dot{y}_{k(t),i}(t) = 0$, (10) implies

$$\lim_{t\to\infty} C_{k(t)}(t) \,|\, X_i(t) \quad y_{k(t),i}(t)| = 0. \tag{29}$$

By Lemma 4, each $C_k(t)$ is bounded from below by a positive constant. Thus (29) implies

$$\lim_{t\to\infty}(X_i(t) - y_i(t)) = 0. \tag{30}$$

Similarly,

$$\lim_{t\to\infty}(X_i(t) - Y_i(t)) \quad 0, \tag{31}$$

and by (30) and (31) together

$$\lim_{t\to\infty}(Y_i(t) - y_i(t)) = 0,$$

or $\epsilon_i \quad 0$, which completes the proof.

5. Stability Properties are Graded in the Time Lag τ

We consider now the case of Theorem 1 which has a prediction theoretic interpretation; namely, we require that for each $\tau \geqslant 0$ the outputs $x_i(t) \to 0$ as $t \to \infty$ if no inputs occur. This case is characterized by the inequalities $\alpha > \beta > 0$.

PROPOSITION 2. If $\alpha > \beta > 0$, then $\sigma(\tau)$ is monotone increasing in $\tau \geqslant 0$, and $\sigma(0) = \sigma \equiv u + 2(\beta - \alpha)$.

PROOF. By Lemma 2, for any fixed $\tau \geqslant 0$, the zero $s(\tau)$ of largest real part of $R_\tau(s) = s + \alpha - \beta e^{-\tau s}$ is real. The proof of Lemma 2 shows also that $s(\tau) < 0$ whenever $\alpha > \beta > 0$. Thus

$$-| s(\tau)| + \alpha = \beta e^{\tau | s(\tau)|}. \tag{32}$$

Suppose any two nonnegative values τ_1 and τ_2 of τ are given such that

$$| s(\tau_2)| > | s(\tau_1)| \geqslant 0.$$

Then by (32),

$$\beta(e^{\tau_1 | s(\tau_1)|} - e^{\tau_2 | s(\tau_2)|}) = | s(\tau_2)| \qquad s(\tau_1)| > 0.$$

Since $\beta > 0$,

$$e^{\tau_1 | s(\tau_1)|} > e^{\tau_2 | s(\tau_2)|}$$

and thus

$$\tau_1 | s(\tau_1)| > \tau_2 | s(\tau_2)| \geqslant 0. \tag{33}$$

In particular, $\tau_1 > 0$. Since $\tau_1 > 0$ and $| s(\tau_2)| > | s(\tau_1)|$,

$$\tau_1 | s(\tau_2)| > \tau_1 | s(\tau_1)|. \tag{34}$$

(34) along with (33) implies

$$\tau_1 | s(\tau_2)| > \tau_2 | s(\tau_2)|$$

Since $| s(\tau_2)| > 0$,

$$\tau_1 > \tau_2 .$$

We have hereby shown that $\tau_1 \leqslant \tau_2$ implies $| s(\tau_2)| \leqslant | s(\tau_1)|$. Since $\sigma(\tau) = u + 2s(\tau) = u - 2 | s(\tau)|$, $\tau_1 \leqslant \tau_2$ implies $\sigma(\tau_1) \leqslant \sigma(\tau_2)$. $\sigma(\tau)$ is therefore a monotone increasing function of τ, for $\tau \geqslant 0$.

$s(0)$ satisfies the equation

$$R_0(s) = s + \alpha - \beta = 0.$$

Thus $s(0) = \beta - \alpha$ and

$$\begin{aligned} \sigma(0) &= u - 2 | s(0)| \\ &= u - 2 | \beta - \alpha \\ &= u - 2(\alpha - \beta) \\ &= u + 2(\beta - \alpha) \\ &= \sigma. \end{aligned}$$

Proposition 2 shows that if $\alpha > \beta > 0$ and $\sigma(\tau_0) > 0$ for some $\tau_0 \geqslant 0$, then Theorem 1 holds for all $\tau \geqslant \tau_0$ and $n \geqslant 2$. We therefore say that the stability properties of (10) and (18) are *graded* in the time lag τ. In particular, if $u > 2(\alpha - \beta) > 0$, then Theorem 1 holds for all $\tau \geqslant 0$ and $n \geqslant 2$. Thus the condition needed to guarantee convergence of the probabilities $X_i(t)$ and $y_{ki}(t)$ to the limiting equations $Q_i = P_{ki}$ as $t \to \infty$ becomes *weaker* as τ increases if also the outputs $x_i(t)$ eventually decay to zero for all $\tau \geqslant 0$.

This gradation of stability properties with respect to τ can be heuristically interpreted if we think of (10) and (18) as the description of a flow over a graph as in the Introduction. Let each edge e_{ij} of the graph associated with (10) and (18) have a length, which we take to be 1 for all edges. The time lag τ can then be interpreted as the inverse velocity $1/v$ of the flows along all the edges. Theorem 1 says that if the probabilities have limits of the form $Q_i = P_{ki}$ when the flow velocity is v_0, then they have limits of this form also for all smaller flow velocities. If we consider the limits $Q_i = P_{ki}$ to be the "stable" or "equilibrium" phase of (10) and (18), and regard the velocity of the flow as an indicator of the "strength" of the interaction between vertices, then Proposition 2 says it gets harder to guarantee the stability of this flow as the interaction strength gets stronger. This fact is intuitively plausible.

The fact that we can guarantee stability for all flow velocities if $u > 2(\alpha - \beta) > 0$ has the following interpretation. The parameters α, β, and u can be thought of as characterizing the materials which go into the construction of each separate vertex and each separate edge of (10) and (18). From this point of view, the parameters α, β, and u are "local" quantities, since they do not take into consideration the various ways in which the vertices and edges can interact. In constructing these vertices and edges, it is natural to ask the following question: can we choose our materials once and for all in such a way that (*) will eventually be stable no matter how strongly the vertices and edges interact? Theorem 1 and Proposition 2 guarantee that the answer to this question is "yes" because $\sigma(0)$ is independent of $\tau \geqslant 0$.

6. THE CASE $\sigma(\tau) < 0$

The condition $\sigma(\tau) > 0$ is not superfluous to guaranteeing the limiting equations $Q_i = P_{ki}$ of Theorem 1. We illustrate this fact in the case $\tau = 0$ for simplicity.

PROPOSITION 3 Suppose $\sigma < 0$. Then

$$y_{jk}(t) \quad y_{jk}(0)| \leqslant 2 \log \left(1 + \frac{\beta x^2(0)}{| \sigma | z^{(j)}(0)}\right.$$

for all t 0.

We can thus make the deviation of P_{jk} from $y_{jk}(0)$ as small as we please by choosing $\sigma < 0$ and $|\sigma|$ sufficiently large. In particular, if

$$y_{jk}(0) \quad y_{ik}(0) \quad 2\log\left(1 + \frac{\beta x^2(0)}{\sigma \mid z^{(j)}(0)}\right)\left(1 + \frac{\beta x^2(0)}{\mid \sigma \mid z^{(i)}(0)}\right)$$

then the equations $Q_k = P_{jk}$ and $Q_k = P_{ik}$ cannot be simultaneously fulfilled.

The proof of this Proposition is contained in [2].

7 Prediction and Learning Theoretic Remarks

By Theorem 1, $X_i(0) = y_i(0) = Y_i(0)$ implies $X_i(t) = y_i(t) = Y_i(t) =$ constant for all $t \geq 0$, and in any case $X_i(t)$ and all $y_{ki}(t)$ lie in $[m_i, M_i]$ for all $t \geq 0$. In the former situation, we say that the complete graph with loops "remembers" its initial data with perfect accuracy, and in the latter case that it remembers its initial data within an error of $M_i - m_i$.

These facts are direct analogs of a property found in an outstar with an input-free border [1]; namely, the outstar probabilities

$$y_{1i}(t) = z_{1i}(t)\left[\sum_{m=2}^{n} z_{1m}(t)\right]^{-1}$$

and

$$X_i(t) = x_i(t)\left[\sum_{m=2}^{n} x_m(t)\right]^{-1},$$

$i = 2,\ldots, n$, are constant for all $t \geq 0$ if $X_i(0) = y_{1i}(0)$, and in any case they always lie in the smallest interval that contains $X_i(0)$ and $y_{1i}(0)$.

This analogy between the memory of an input-free complete graph with loops and of an input-free outstar is not, however, complete. For example, in the complete graph with loops the limiting equations $Q_i = P_{ki}$ cannot be guaranteed unless $\sigma(\tau) > 0$, whereas the analogous limiting equations $Q_i = P_{1i}$ in an outstar held for all values of $\sigma(\tau)$. The analogy breaks down still further, as our next theorem will show.

This theorem studies the following question: given any probability distribution $\{\theta_i : i = 1, 2,\ldots, n\}$, can inputs be found which in finite time bring $X_i(t)$, $y_i(t)$, and $Y_i(t)$ within an interval of prescribed smallness enclosing θ_i? That is, can the experimenter E find an experiment which in finite time "teaches" the machine M the probabilities θ_i to within an arbitrarily small error? The answer is "yes." Once the inputs cease, Theorem 1 guarantees that M remembers the probabilities to within the same error for all future times just so long as no new inputs occur. This experiment is analogous to the experiment performed on an outstar which teaches $X_i(t)$ and $y_{1i}(t)$ the probabilities θ_i, $i = 2,\ldots, n$, to within an arbitrarily small error.

The analogy between complete graphs with loops and outstars breaks down in "recall" experiments in which E presents a test input to one vertex only and measures the output produced thereby to test the memory of E. In an outstar, such a recall experiment does not alter the accuracy of M's memory of earlier teaching experiments. In a complete graph with loops, the very act of recall helps to destroy the memory of earlier teaching experiments. The complete graph with loops must therefore be regularly retaught after recall experiments, whereas the outstar need never be retaught.

A previous paper [2] studied a complete graph *without* loops which differs from both the graphs previously discussed in that it forgets its initial data even when $\sigma > 0$. These three examples illustrate the profound effect which the coefficient matrix P—that is, the "geometry" of the graph—has on flow dynamics, and in particular on the "memory" of M.

8 Graphs which are Not Eventually Input-Free

We now define inputs which can teach M any probability distribution to a fixed degree of accuracy within finite time. Our result discusses the global limits of X_i and y_{jk}, and the oscillations of X_i relative to the functions $Y_{i,\theta} = \max\{Y_i, \theta_i\}$ and $y_{i,\theta} = \min\{y_i, \theta_i\}$.

Theorem 2. *Let (*) be given with any fixed $n \geq 2$ and any $\tau \geq 0$ such that $s(\tau) < 0$ and $\sigma_1(\tau) \equiv u + s(\tau) > 0$. Define the inputs*

$$I_i(t) = \theta_i I(t),$$

where $\{\theta_j : j = 1, 2,\ldots, n\}$ is a fixed, but arbitrary, probability distribution, and I is any bounded, continuous, and nonnegative function for which positive constants k and T_0 exist such that

$$\int^{t} e^{u v} I(v)\, dv \quad k e^{u t} \qquad T_0 . \tag{36}$$

Then for arbitrary continuous and nonnegative initial data

(1) (*limiting behavior*) *the limits*

$$Q_i \quad \lim_{t\to\infty} X_i(t) \qquad \text{and} \qquad P_{jk} \quad \lim_{t\to\infty} y_{jk}(t)$$

exist and satisfy the equations

$$Q_i = P_{ki} = \theta_i , \tag{37}$$

$i, k = 1, 2,..., n.$

(2) (*oscillatory behavior*) *the functions* $\dot{y}_{i,\theta}$, $\dot{Y}_{i,\theta}$, $X_i - y_{i,\theta}$, *and* $X_i - Y_{i,\theta}$ *change sign at most once and not at all if* $y_i(0) \leqslant X_i(0) \leqslant Y_i(0)$.

As in Theorem 1, the assertions concerning oscillations do not depend on the sign of $\sigma_1(\tau)$ or $s(\tau)$.

LEMMA 7. *For any values of* $\sigma_1(\tau)$ *and* $s(\tau)$, *the functions* $\dot{y}_{i,\theta}$, $\dot{Y}_{i,\theta}$, $X_i - y_{i,\theta}$, *and* $X_i - Y_{i,\theta}$ *change sign at most once, and not at all if* $y_i(0) \leqslant X_i(0) \leqslant Y_i(0)$.

PROOF. Let $X_i^{(\theta)} = X_i - \theta_i$, $y_{jk}^{(\theta)} = y_{jk} - \theta_j$, $y_i^{(\theta)} = y_i - \theta_i$, and $Y_i^{(\theta)} = Y_i - \theta_i$. Then (9) and (10) become

$$\dot{X}_i^{(\theta)} = A \sum_{m=1}^{n} X_m(t - \tau)[y_{mi}^{(\theta)} - X_i^{(\theta)}] - BX_i^{(\theta)}, \tag{9}$$

and

$$\dot{y}_{jk}^{(\theta)} = C_j(X_k^{(\theta)} - y_{jk}^{(\theta)}) \tag{10}$$

From these equations and the nonnegativity of A, B, and C_j, the following alternatives are apparent by inspection.

CASE 1. If $X_i^{(\theta)}(t_0) \geqslant 0$ and $y_i^{(\theta)}(t_0) \geqslant 0$, then $X_i^{(\theta)}(t) \geqslant 0$ and $y_i^{(\theta)}(t) \geqslant 0$ for $t \geqslant t_0$. If moreover $X_i^{(\theta)}(t_0) \leqslant Y_i^{(\theta)}(t_0)$, then $X_i^{(\theta)}(t) \leqslant Y_i^{(\theta)}(t)$ and $Y_i^{(\theta)}(t)$ is monotone decreasing for $t \geqslant t_0$. On the other hand, if $X_i^{(\theta)}(t_0) > Y_i^{(\theta)}(t_0)$, then $X_i^{(\theta)}(t)$ is monotone decreasing and all $y_{ki}^{(\theta)}(t)$ are monotone increasing until the first time $t = t_1$, at which $X_i^{(\theta)}(t) = Y_i^{(\theta)}(t)$. If no such time exists, all limits Q_i and P_{ki} exist and $Q_i \geqslant P_{ki} \geqslant \theta_i$. If such a time does exist, the preceding case holds for all $t \geqslant t_1$.

CASE 2. If $X_i^{(\theta)}(t_0) \leqslant 0$ and $Y_i^{(\theta)}(t_0) \leqslant 0$, then the arguments of Case 1 go through with inequalities reversed, and $y_i^{(\theta)}$ and $Y_i^{(\theta)}$ interchanged. Thus either all limits Q_i and P_{ki} exist, or there is a t_1, such that $y_i^{(\theta)}(t) \leqslant X_i^{(\theta)}(t)$ for $t \geqslant t_1$.

CASE 3. If $Y_i^{(\theta)}(0) \geqslant 0 \geqslant y_i^{(\theta)}(0)$ and $Y_i^{(\theta)}(0) > y_i^{(\theta)}(0)$, then either $Y_i^{(\theta)}(t) \geqslant 0 \geqslant y_i^{(\theta)}(t)$ and $Y_i^{(\theta)}(t) > y_i^{(\theta)}(t)$ for all $t \geqslant 0$, or we eventually enter either Case 1 or Case 2. Suppose that the former alternative occurs. If moreover $X_i^{(\theta)}(0) \notin [y_i^{(\theta)}(0), Y_i^{(\theta)}(0)]$, then $X_i^{(\theta)}(t)$ and all $y_{ki}^{(\theta)}(t)$ are monotonic until the first time $t = t_2$ at which $X_i^{(\theta)}(t) \in [y_i^{(\theta)}(t), Y_i^{(\theta)}(t)]$.

Thereafter, $X_i^{(\theta)}(t) \in [y_i^{(\theta)}(t), Y_i^{(\theta)}(t)]$ and $Y_i^{(\theta)}(t)$ is monotone decreasing whereas $y_i^{(\theta)}(t)$ is monotone increasing. Both limits $Y_i(\infty) = \lim_{t\to\infty} Y_i(t)$ and $y_i(\infty) = \lim_{t\to\infty} y_i(t)$ therefore exist. If $Y_i(\infty) = y_i(\infty)$, all limits Q_i and P_{ki} exist and are equal.

Case 1–3 exhaust all alternatives, and thus the assertions of the Lemma are apparent.

Lemma 7 suffices to prove Theorem 2 when all θ_i are positive once the following information concerning $x = \sum_{k=1}^{n} x_k$ is made available.

LEMMA 8 *There exist positive constants* λ_i, $i = 1, 2, 3, 4$, *such that for* $t > \tau$,

$$\lambda_1 + \lambda_2 \int_\tau^t e^{-s(\tau)v} I(v)\, dv \leqslant x(t)\, e^{-s(\tau)t} \leqslant \lambda_3 + \lambda_4 \int_\tau^t e^{-s(\tau)v} I(v)\, dv. \tag{38}$$

PROOF. Since x obeys (15), we can apply the following representation theorem ([6], pp. 73–75): for $t > 0$,

$$x(t) = x(0)\, k(t) + \int_0^\tau [\dot{x}(v) + \alpha x(v)]\, k(t - v)\, dv + \int_\tau^t I(v)\, k(t - v)\, dv, \tag{39}$$

where $k(t)$ is the unique function satisfying

(a) $k(t) = 0, \quad t < 0,$

(b) $k(0) = 1,$

(c) $k(t)$ is continuous on $[0, \infty]$,

(d) $k(t)$ satisfies the equation

$$\dot{k}(t) = -\alpha k(t) + \beta k(t - \tau).$$

By Lemma 3, $k(t)$ can be written in the form

$$k(t) = e^{s(\tau)t}[c + e^{-mt} H(t)] \tag{40}$$

for $t > \tau$, where c and m are positive and H is bounded and nonnegative. The proof is completed by substituting (40) into (39), rearranging terms, and using the nonnegativity of all quantities to make the now obvious estimates.

REMARK. The limits $Q_i = P_{ki} = \theta_i$ can be derived from Lemmas 7 and 8 just as in Theorem 1 if all θ_i are positive. This is because essentially the same boundedness estimates can be made on the coefficients A, B, C_j, and their derivatives in Theorem 2 as we made on A and C_j in Theorem 1. For example, by Lemma 7

$$X_i(t) \geqslant \min\{X_i(0), y_i(0), \theta_i\},$$

which is positive if θ_i is positive except possibly for a finite amount of time if $X_i(0) = 0$. This fact allows us to bound C_i and $|\dot{C}_i|$ from above and below by positive constants, as in the proof of Theorem 1.

Even when at least one $\theta_i = 0$, Lemma 8 allows us to conclude that $Q_i = P_{ki} = \theta_i$ in several cases. For example, consider the subcase of Case 1 for which $Q_i \geq P_{ki} \geq \theta_i$. If $Q_i = \theta_i$, we are done. Suppose not. Then by (9) there is a T such that $t \geq T$ implies

$$|\dot{X}_i(t)| \geq \tfrac{1}{2}(Q_i - \theta_i)\, B(t).$$

Thus by Lemma 8,

$$|\dot{X}_i(t)| \geq \tfrac{1}{2}(Q_i - \theta_i)\frac{I e^{-s(\tau)}}{\lambda_3 + \lambda_4 \int_\tau^t I(v)\, e^{-s(\tau)v}\, dv}$$

$$= \frac{1}{2\lambda_4}(Q_i - \theta_i)\frac{d}{dt}\log\left(\lambda_3 + \lambda_4 \int_\tau^t I(v)\, e^{-s(\tau)v}\, dv\right)$$

which in integral form is

$$X_i(T) \geq X_i(t) + \frac{1}{2\lambda_4}(Q_i - \theta_i)\log\left(\frac{\lambda_3 + \lambda_4 \int_\tau^t I(v)\, e^{-s(\tau)v}\, dv}{\lambda_3 + \lambda_4 \int_\tau^T I(v)\, e^{-s(\tau)v}\, dv}\right)$$

since $|\dot{X}_i(t)| = -\dot{X}_i(t)$ for $t \geq T$.

By the hypothesis that $s(\tau) < 0$,

$$\alpha - |s(\tau)| = \beta e^{\tau|s(\tau)|} > 0,$$

or $\alpha > |s(\tau)|$. Thus by (36),

$$\int_\tau^t e^{s(\tau)(t-v)} I(v)\, dv \geq \int e^{-\alpha(t-v)} I(v)\, dv$$

$$\geq k$$

for $t > T_0$, and

$$X_i(T) \geq X_i(t) - \frac{1}{2\lambda_4}(Q_i - \theta_i)\log\left(\frac{\lambda_3 + \lambda_4 k e^{|s(\tau)|t}}{\lambda_3 + \lambda_4 \int_\tau^T I(v)\, e^{-s(\tau)v}\, dv}\right)$$

for $t \geq T$. Since $Q_i > \theta_i$, $\lim_{t\to\infty} X_i(t) = -\infty$, which contradicts the nonnegativity of X_i, so that $\theta_i = Q_i = P_{ki}$. The corresponding subcase of Case 2 is treated analogously, and the subcase of Case 3 for which $Y_i(\infty) = y_i(\infty)$ is already proved. All the following estimates are aimed at treating the remaining subcases for probability distributions with at least one zero entry.

Throughout the remainder of the proof, we will consider only Case 1, since Cases 2 and 3 can be treated by obvious modifications. We will need the following estimates of $X_i^{(\theta)}$ from below and above.

LEMMA 9. *If $X_i^{(\theta)}(t_0) \geq 0$ and $y_i^{(\theta)}(t_0) > 0$, then there exists a positive constant $\xi_i^{(\theta)}$ such that*

$$X_i^{(\theta)}(t) \geq \xi_i^{(\theta)} e^{s(\tau)t} \qquad t \geq t_0 \tag{41}$$

PROOF. The proof consists in showing that the solution $w_i^{(\theta)}$ of the following system is a minorante for $X_i^{(\theta)}$ when $t \geq t_0$.

$$\dot{w}_i^{(\theta)} = A(v_i^{(\theta)} - w_i^{(\theta)})\,\chi(v_i^{(\theta)} - w_i^{(\theta)}) - Bw_i^{(\theta)} \tag{42}$$

and

$$\dot{v}_i^{(\theta)} = C_i(w_i^{(\theta)} - v_i^{(\theta)}), \tag{43}$$

where

$$\chi(w) = \begin{cases} 0 & w \geq 0 \\ 1 & w < 0, \end{cases}$$

$w_i^{(\theta)}(t_0) = X_i^{(\theta)}(t_0)$, and $v_i^{(\theta)}(t_0) = y_i^{(\theta)}(t_0)$. The coefficients A, B, and C_i have their usual meaning.

By Lemma 7, $X_i^{(\theta)}(t) \geq 0$ and $y_i^{(\theta)}(t) \geq 0$ for $t \geq t_0$. Moreover all $X_m(t - \tau)$ are always nonnegative. Thus for $t \geq t_0$,

$$\begin{aligned} \dot{X}_i^{(\theta)} &= A \sum_{m=1}^{n} X_m(t-\tau)[y_{mi}^{(\theta)} - X_i^{(\theta)}] - BX_i^{(\theta)} \\ &\geq A(y_i^{(\theta)} - X_i^{(\theta)}) - BX_i^{(\theta)} \\ &\geq A(y_i^{(\theta)} - X_i^{(\theta)})\,\chi(y_i^{(\theta)} - X_i^{(\theta)}) - BX_i^{(\theta)}. \end{aligned} \tag{44}$$

It readily follows that $X_i^{(\theta)}(t) \geq w_i^{(\theta)}(t)$ for $t \geq t_0$ by comparing (44) with (42), and noting that a decrease in $w_i^{(\theta)}$ can only cause a decrease in $v_i^{(\theta)}$.

Consider (43). If $v_i^{(\theta)}(t_0) < w_i^{(\theta)}(t_0)$, then $v_i^{(\theta)}$ increases until the first time $t = T > t_0$ at which $v_i^{(\theta)}(t) = w_i^{(\theta)}(t)$. If no such T exists, then for all $t \geq t_0$,

$$X_i^{(\theta)}(t) \geq w_i^{(\theta)}(t) \geq v_i^{(\theta)}(t) \geq v_i^{(\theta)}(t_0) \geq v_i^{(\theta)}(t_0)\, e^{s(\tau)(t-t_0)},$$

which completes the proof in this case. If such a T does exist, then

$$\dot{v}_i^{(\theta)}(T) = 0 \qquad \text{whereas} \qquad \dot{w}_i^{(\theta)}(T) = -B(T)\, w_i^{(\theta)}(T) < 0.$$

Hence

$$\chi(v_i^{(\theta)}(t) \quad w_i^{(\theta)}(t)) = 0, \qquad t \geq T,$$

so that

$$\dot{w}_i^{(\theta)}(t) = -B(t)\, w_i^{(\theta)}(t), \qquad t \geq T.$$

As in the previous Remark, we therefore find

$$w_i^{(\theta)}(t) \geq w_i^{(\theta)}(T)\, e^{-\int_T^t B dv}$$

$$\geq \frac{w_i^{(\theta)}(T)(\lambda_3 + \lambda_4 \int_\tau^T I(v)\, e^{-s(\tau)v}\, dv)}{\lambda_3 + \lambda_4 \int_\tau^t I(v)\, e^{-s(\tau)v}\, dv}$$

Since $I \quad \sup\{I(t) : t \geq 0\}$ is finite,

$$w_i^{(\theta)}(t) \geq \eta_i^{(\theta)}(t)\, e^{s(\tau)t}$$

where

$$\eta_i^{(\theta)}(t) = \frac{w_i^{(\theta)}(T)(\lambda_3 + \lambda_4 \int_\tau^T I(v)\, e^{-s(\tau)v}\, dv)}{\left(\lambda_3 + \frac{\lambda_4 I e^{-s(\tau)t}}{s(\tau)}\right) e^{s(\tau)t} + \frac{\lambda_4 I}{|s(\tau)|}}$$

Since $\eta_i^{(\theta)}$ is positive and has a positive limit as $t \to \infty$, the proof is complete

LEMMA 10. *Suppose $Y_i^{(\theta)}(t_0) \geq X_i^{(\theta)}(t_0) \geq 0$ and $y_i^{(\theta)}(t_0) \geq 0$, where we can choose $t_0 \geq T_0$ without loss of generality. Then there exists a $\mu \in (0, 1)$ and a $T_1 = T_1(\mu)$ such that*

$$X_i^{(\theta)}(t) \leq (1 - \mu) \quad {}_i^{(\theta)}(t - T_1) \tag{46}$$

for every $t \geq t_0 + T_1$.

PROOF. Proceeding as in Lemma 9, we define for $t \geq t_0$ a majorante $W_i^{(\theta)}$ of $X_i^{(\theta)}$ by the equation

$$\dot{W}_i^{(\theta)} = A(Y_i^{(\theta)} - W_i^{(\theta)}) - BW_i^{(\theta)}, \tag{47}$$

where A, B, and $Y_i^{(\theta)}$ have their usual meaning, and $W_i^{(\theta)}(t_0) = X_i^{(\theta)}(t_0)$

Integrating (47) in $[t_1, t]$ yields

$$W_i^{(\theta)}(t, t_1) = U_i^{(\theta)}(t, t_1) + V_i^{(\theta)}(t, t_1) \tag{48}$$

where

$$U_i^{(\theta)}(t, t_1) = W_i^{(\theta)}(t_1)\, Z^{-1}(t, t_1), \tag{49}$$

$$V_i^{(\theta)}(t, t_1) = Z^{-1}(t, t_1) \int_{t_1}^t Y_i^{(\theta)} AZ(v, t_1)\, dv, \tag{50}$$

and

$$Z(t, t_1) = \exp\left[\int_{t_1}^t (A + B)\, dw\right]. \tag{51}$$

Since $W_i^{(\theta)}(t_1) \leq Y_i^{(\theta)}(t_1)$ for $t_1 \geq t_0$, (49) implies

$$U_i^{(\theta)}(t, t_1) \leq Y_i^{(\theta)}(t_1)\, Z^{-1}(t, t_1). \tag{52}$$

To evaluate (50) recall that

$$A + B = \frac{\beta x(t - \tau) \pm}{x}$$

$$\frac{d}{dt} \log x + \alpha$$

and so

$$Z(t, t_1) = \frac{x(t)\, e^{\alpha t}}{x(t_1)\, e^{\alpha t_1}}$$

(50) can now be written as

$$V_i^{(\theta)}(t, t_1) = \frac{}{x(t)\, e^{\alpha t}} \int_{t_1}^t Y_i^{(\theta)} Axe^{\alpha v}\, dv$$

and since $0 \leq Y_i^{(\theta)}(v) \leq Y_i^{(\theta)}(t_1)$ for $v \geq t_1$,

$${}_i^{(\theta)}(t, t_1) \leq {}_i^{(\theta)}(t_1)\, R(t, t_1) \tag{53}$$

where

$$R(t, t_1) = \frac{1}{x(t)\, e^{\alpha t}} \int_{t_1}^t Axe^{\alpha v}\, dv$$

Recalling that $A(v) = [\beta x(v - \tau)/x(v)]$, we find

$$R(t, t_1) = \frac{}{x(t)\, e^{\alpha t}} \quad {}_{t_1} \beta x(v \quad \tau)\, e^{\alpha v}\, dv$$

and since

$$\beta x(v \quad \tau)\, e^{\alpha v} \quad e^{\alpha v}(\dot{x} + \alpha x - I)$$
$$\frac{d}{dv}(x e^{\alpha v}) - I e^{\alpha v},$$
$$R(t, t_1) = \quad - Z^{-1}(t, t_1) - \frac{1}{x(t)\, e^{\alpha t}} \int_{t_1}^{t} I e^{\alpha v}\, dv. \tag{54}$$

We now combine (48), (52), (53), and (54) to find that

$$X_i^{(0)}(t) \leqslant W_i^{(0)}(t) \leqslant Y_i^{(0)}(t_1)\, P(t, t_1), \tag{55}$$

where

$$P(t, t_1) = \qquad \frac{1}{x(t)\, e^{\alpha t}} \int_{t_1}^{t} I e^{\alpha v}\, dv \tag{56}$$

By (36) and (38),

$$P(t, t_1) \leqslant \qquad \frac{1}{\lambda_3 + \lambda_4 I \alpha^{-1}} \int \qquad {}^{\alpha(t-v)} I(v)\, dv.$$

It remains only to estimate $\int_{t_1}^{t} e^{-\alpha(t-v)} I(v)\, dv$. By (36),

$$k \leqslant \int_{\tau}^{t} e^{-\alpha(t-v)} I(v)\, dv \leqslant \frac{I}{\alpha}$$

for $t \geqslant T_0$. In particular, for $t \geqslant t_1$ $(\geqslant T_0)$,

$$\int_{t_1}^{t} e^{-\alpha(t-v)} I(v)\, dv \geqslant k - \int_{\tau}^{t_1} e^{-\alpha(t-v)} I(v)\, dv$$
$$\geqslant k - \frac{I}{\alpha} e^{-\alpha(t-t_1)},$$

and there surely exists a T_1 such that for $t \quad t_1 \quad T_1$

$$\int_{t_1}^{t} e^{-\alpha(t-v)} I(v)\, dv \qquad \frac{k}{2}$$

Thus for $t \qquad T_1$

$$P(t, t_1) \qquad - \mu,$$

where $\mu = [k/2(\lambda_3 + \lambda_4 I \alpha^{-1})]$, which along with (55) completes the proof.

The last lemma which we will need represents $y_{jk}^{(0)}$ in integral form.

Lemma 11 *$y_{jk}^{(0)}$ can be expressed in integral form as*

$$y_{jk}^{(0)} \qquad \frac{y_{jk}^{(0)}(0) + k_j \int_0^t X_j(v - \tau)\, X_k^{(0)}(v)\, N(v)\, dv}{1 + k_j \int_0^t X_j(v - \tau)\, N(v)\, dv} \tag{57}$$

where $k_j \quad 1/\beta \sum_{m=1}^{n} z_{jm}(0)$ *and*

$$N(v) = e^{uv} x(v - \tau)\, x(v).$$

Proof. Integrate (10). Then

$$y_{jk}(t) = e^{-\int_0^t C_j dw} \left[y_{jk}(0) + \int_0^t X_k C_j e^{\int_0^v C_j dw}\, dv \right]. \tag{58}$$

Write (14) in the form

$$C_j(t) = \frac{d}{dt} \log \left[k_j + \int_0^t X_j(v \quad \tau)\, N(v)\, dv \right]$$

and substitute into (58). Then

$$y_{jk}(t) = \frac{y_{jk}(0) + k_j \int_0^t X_j(v - \tau)\, X_k(v)\, N(v)\, dv}{1 + k_j \int_0^t X_j(v - \tau)\, N(v)\, dv}$$

Subtract θ_k and find (57).

9. Proof of Theorem 2

We consider only Case 1, so that $X_i^{(0)}(t) \geqslant 0$ and $y_i^{(0)}(t) \geqslant 0$ for $t \geqslant t_0$. We assume that $X_i^{(0)}(t) \leqslant Y_i^{(0)}(t)$ for $t \geqslant t_0$, since otherwise all limits exist and have the correct distribution. We also let $t_0 = T_0 = 0$ for convenience of exposition. Then by (10), $Y_i^{(0)}(t)$ is monotone decreasing, the limit $Y_i^{(0)}(\infty) = \lim_{t\to\infty} Y_i^{(0)}(t)$ exists, and $Y_i^{(0)}(\infty) \geqslant 0$. If $Y_i^{(0)}(\infty) = 0$, then all limits Q_i and P_{ki} exist and equal θ_i. It remains only to consider the case $Y_i^{(0)}(\infty) > 0$. The proof proceeds by showing first that in this case the limit $\lim_{t\to\infty} X_i^{(0)}(t)$ exists and equals $Y_i^{(0)}(\infty) > 0$. This fact is then used to show that all limits $\lim_{t\to\infty} y_{ki}^{(0)}(t)$ exist and also equal $Y_i^{(0)}(\infty)$. Then we can draw the contradiction that $\lim_{t\to\infty} X_i^{(0)}(t) = 0$, from which we conclude that $Y_i^{(0)}(\infty) = 0$ after all in Case 1, and thus that $Q_i = P_{ki} = \theta_i$ in this Case. Analogous arguments are then readily seen to hold in Cases 2 and 3.

(I) We now prove that $\lim_{t\to\infty} X_i^{(\theta)}(t) = Y_i^{(\theta)}(\infty)$, where we suppose $Y_i^{(\theta)}(\infty) > 0$. Let $K(t)$ be that integer such that $Y_i^{(\theta)}(t) = y_{K(t),i}^{(\theta)}(t)$. Suppose that $K(t)$ takes on the values $r_1, r_2, \ldots, r_m, \ldots$ in the intervals $[0, T_1)$, $[T_1, T_2), \ldots, [T_{m-1}, T_m), \ldots$, respectively. Then by (57),

$$Y_i^{(\theta)}(t) = \frac{y_{r_m,i}^{(\theta)}(0) + k_{r_m} \int_0^t X_{r_m}(v-\tau) X_i^{(\theta)}(v) N(v)\, dv}{1 + k_{r_m} \int_0^t X_{r_m}(v-\tau) N(v)\, dv}$$

for $t \in [T_{m-1}, T_m)$. Since $Y_i^{(\theta)}$ is monotone decreasing and $Y_i^{(\theta)}(\infty) > 0$,

$$Y_i^{(\theta)}(\infty) \left[1 + k_{r_m} \int_0^t X_{r_m}(v-\tau) N(v)\, dv\right] \leqslant y_{r_m,i}^{(\theta)}(0) + k_{r_m} \int_0^t X_{r_m}(v-\tau) X_i^{(\theta)}(v) N(v)\, dv. \tag{59}$$

Again by the monotone decrease of $Y_i^{(\theta)}$ to $Y_i^{(\theta)}(\infty)$, we can find for every $\epsilon > 0$ a t_ϵ such that $Y_i^{(\theta)}(t) \leqslant Y_i^{(\theta)}(\infty) + \epsilon$ for $t \geqslant t_\epsilon$. We will consider in particular only ϵ's with $0 < \epsilon \leqslant (\mu/1-\mu)\, Y_i^{(\theta)}(\infty)$, where μ is defined in Lemma 10. We now estimate the integral in the right-hand side of (59) in terms of any such ϵ and the functions

$$H_{i,\delta}^{(\theta)}(v) = \begin{cases} 1 & \text{if } X_i^{(\theta)}(v) \leqslant \delta \\ 0 & \text{if } X_i^{(\theta)}(v) > \delta, \end{cases}$$

and $J_{i,\delta}^{(\theta)} = 1 - H_{i,\delta}^{(\theta)}$, which we define for every fixed $\delta > 0$

By Lemma 10 we find that for every $t \geqslant t_\epsilon + T_1$,

$$\leqslant \int_0^{t_\epsilon+T_1} X_{r_m}(v-\tau) X_i^{(\theta)}(v) N(v)\, dv + \delta \int_{t_\epsilon+T_1}^t X_{r_m}(v-\tau) H_{i,\delta}^{(\theta)}(v) N(v)\, dv$$
$$+ (1-\mu) \int_{t_\epsilon+T_1}^t X_{r_m}(v-\tau) J_{i,\delta}^{(\theta)}(v)\, Y_i^{(\theta)}(v-T_1) N(v)\, dv$$

Since $Y_i^{(\theta)}(v - T_1) \leqslant Y_i^{(\theta)}(\infty) + \epsilon$ for $v - T_1 \geqslant t_\epsilon$, and $\epsilon \leqslant (\mu/1-\mu)\, Y_i^{(\theta)}(\infty)$

$$\int_{t_\epsilon+T_1}^t X_{r_m}(v-\tau) X_i^{(\theta)}(v) N(v)\, dv \leqslant \delta \int_{t_\epsilon+T_1}^t X_{r_m}(v-\tau) H_{i,\delta}^{(\theta)}(v) N(v)\, dv$$
$$+ Y_i^{(\theta)}(\infty) \int_{t_\epsilon+T_1}^t X_{r_m}(v-\tau) J_{i,\delta}^{(\theta)}(v) N(v)\, dv. \tag{60}$$

Substituting (60) into (59) and rearranging terms, we find for any fixed ϵ in $(0, (\mu/1-\mu)\, Y_i^{(\theta)}(\infty))$ that

$$\int_{t_\epsilon+T_1}^t X_{r_m}(v-\tau) H_{i,\delta}^{(\theta)}(v) N(v)\, dv$$
$$\leqslant \frac{y_{r_m,i}^{(\theta)}(0) - Y_i^{(\theta)}(\infty)}{k_{r_m}(Y_i^{(\theta)}(\infty) - \delta)} + \frac{1 - Y_i^{(\theta)}(\infty)}{Y_i^{(\theta)}(\infty) - \delta} \int_0^{t_\epsilon+T_1} X_{r_m}(v-\tau) X_i^{(\theta)}(v) N(v)\, dv$$

and thus

$$\int_{t_\epsilon+T_1}^t X_{r_m}(v-\tau) H_{i,\delta}^{(\theta)}(v) N(v)\, dv \leqslant \frac{1 - Y_i^{(\theta)}(\infty)}{Y_i^{(\theta)}(\infty) - \delta} \left[\frac{1}{k} + \int_0^{t_\epsilon+T_1} N(v)\, dv\right], \tag{61}$$

where $k = \min\{k_j : j = 1, 2, \ldots, n\} > 0$.

We now consider the intervals on which $H_{i,\delta}^{(\theta)} = 1$. By (36), (38), and the boundedness of $I(t)$, it is easily seen from (9) that $|\dot{X}_i|$ is bounded. Thus for every $\delta > 0$ which is ever smaller than $X_i^{(\theta)}$, there is an $L_\delta > 0$ such that $H_{i,\delta}^{(\theta)} = 1$ on intervals of at least length L_δ. If no such δ exists, then $X_i^{(\theta)} \equiv 0$, and we are done. For any such δ, we can write the integral on the left-hand side of (61) as a sum of integrals

$$\int_{M_k^{(\delta)}} X_{r_m}(v-\tau) N(v)\, dv \tag{62}$$

over a sequence of $N_\delta(t)$ nonoverlapping intervals $M_1^{(\delta)}, M_2^{(\delta)}, \ldots, M_{N_\delta(t)}^{(\delta)}$ whose length is at least L_δ. We now estimate the size of the integrals in (62).

If $\theta_{r_m} > 0$, then by Lemma 7 there exists a $\gamma_{r_m} > 0$ such that $X_{r_m}(t) \geqslant \gamma_{r_m}$ for all $t \geqslant 0$. If $\theta_{r_m} = 0$, then either such a $\gamma_{r_m} > 0$ again exists, or else we enter Case 1 of Lemma 7 for all large t. In this situation, Lemma 9 applies and thus $X_i^{(\theta)}(t) \geqslant \xi_i^{(\theta)} e^{s(\tau)t}$ for all large t and hence for all $t \geqslant 0$ with perhaps a change in $\xi_i^{(\theta)}$. In all cases, therefore, $X_i^{(\theta)}(t) \geqslant \xi_i^{(\theta)} e^{s(\tau)t}$ for *all* $t \geqslant 0$, and the integral in (62) exceeds

$$\xi_i^{(\theta)} e^{-\tau s(\tau)} \int_{M_k^{(\delta)}} e^{s(\tau)v} N(v)\, dv \tag{63}$$

By (36) and (38) we can, in turn, find a positive constant $w_i^{(\theta)}$ such tha (63) exceeds

$$w_i^{(\theta)} \int_{M_k^{(\delta)}} e^{\sigma_1(\tau)v}\, dv \tag{64}$$

where $\sigma_1(\tau) = u + s(\tau) > 0$, so that the integral in (64) exceeds $w_i^{(\theta)} L_\delta$. We have therefore shown, by (61), that

$$N_\delta(t) \leqslant \frac{1}{w_i^{(\theta)} L_\delta} \Omega_{i,\delta,\epsilon}^{(\theta)}$$

where

$$\Omega_{i,\delta,\epsilon}^{(\theta)} = \frac{-Y_i^{(\theta)}(\infty)}{Y_i^{(\theta)}(\infty) - \delta} \left[\frac{1}{k} + \int_0^{t_\epsilon + T_1} N(v)\, dv\right]$$

In particular, $N_\delta(t)$ is finite for every $t \geqslant t_\epsilon + T_1$ and there exists a time S_δ such that $X_i^{(\theta)}(t) > \delta$ for all $t \geqslant S_\delta$. This is true for every δ which is ever smaller than $X_i^{(\theta)}$ and $Y_i^{(\theta)}(\infty)$. Since $X_i^{(\theta)}$ eventually exceeds all such δ's and $X_i^{(\theta)}(t) \leqslant Y_i^{(\theta)}(t)$ for all $t \geqslant t_0$, we conclude that $\lim_{t\to\infty} X_i^{(\theta)}(t) = Y_i^{(\theta)}(\infty)$.

(II) We can now show that $\lim_{t\to\infty} y_{ki}^{(\theta)}(t) = Y_i^{(\theta)}(\infty)$ for all $k = 1, 2,\ldots, n$. First express $y_{ki}^{(\theta)}$ in integral form as in (57). Subtracting $Y_i^{(\theta)}(\infty)$ from both sides gives

$$y_{ki}^{(\theta)}(t) \quad Y_i^{(\theta)}(\infty) = A_{ki}^{(\theta)}(t) + B_{ki}^{(\theta)}(t),$$

where

$$A_{ki}^{(\theta)}(t) \qquad \frac{}{+ k_k \int_0^t X_k(v - \tau)\, N(v)\, dv}$$

$$B_{ki}^{(\theta)}(t) = \frac{k_k \int_0^t X_k(v - \tau)[X_i^{(\theta)}(v) - Y_i^{(\theta)}(\infty)]\, N(v)\, dv}{1 + k_k \int_0^t X_k(v - \tau)\, N(v)\, dv}$$

By familiar arguments using $\sigma_1(\tau) > 0$, $\lim_{t\to\infty} A_{ki}^{(\theta)}(t) = 0$. It remains only to show that $\lim_{t\to\infty} B_{ki}^{(\theta)} = 0$. Clearly

$$0 \leqslant |\, B_{ki}^{(\theta)}(t)| \leqslant \frac{k_k \int_0^t X_k(v \quad \tau)\, L_i^{(\theta)}(v)\, N(v)\, dv}{1 + k_k \int_0^t X_k(v - \tau)\, N(v)\, dv},$$

where $L_i^{(\theta)}(t) = |\, X_i^{(\theta)}(t) - Y_i^{(\theta)}(\infty)| \to 0$ as $t \to \infty$. Thus for every $\epsilon > 0$ there is a T_ϵ such that $t \geqslant T_\epsilon$ implies $L_i^{(\theta)}(t) \leqslant \epsilon$ and hence

$$0 \leqslant |\, B_{ki}^{(\theta)}(t)| \leqslant \frac{\epsilon k_k \int_{T_\epsilon}^t X_k(v - \tau)\, N(v)\, dv}{+ k_k \int_{T_\epsilon}^t X_k(v - \tau)\, N(v)\, dv} + o(1) \leqslant \epsilon + o(1),$$

which completes the proof.

(III) We now use the fact that

$$\lim_{t\to\infty} X_i^{(\theta)}(t) = \lim_{t\to\infty} y_{ki}^{(\theta)}(t) = Y_i^{(\theta)}(\infty) > 0$$

to draw the contradiction that $\lim_{t\to\infty} X_i^{(\theta)}(t) = 0$, and thus to show that $Y_i^{(\theta)}(\infty) = 0$ after all. This follows immediately if we write $X_i^{(\theta)}(t)$ in integral form as

$$X_i^{(\theta)}(t) = e^{-\int_0^t B\,dw} \left[X_i^{(\theta)}(0) + \int^t \Gamma_i(v)\, e^{\int_0^v B\,dw}\, dv\right],$$

where $\Gamma(t) \equiv A \sum_{m=1}^n X_m(t - \tau)[y_{mi}^{(\theta)} - X_i^{(\theta)}]$ converges to 0 as $t \to \infty$, and then estimate $e^{\int_0^t B\,dw}$ from above and below by expressions of the form $\lambda e^{|s(\tau)|t}$, λ constant, using (36) and (38). We have hereby shown that the limits Q_i and P_{ki} always exist in Case 1 and have the values $Q_i = P_{ki} = \theta_i$. An identical argument can be carried out in Case 2 with all inequalities reversed, and $y_i^{(\theta)}$ and $Y_i^{(\theta)}$ interchanged. In Case 3 we need deal only with the situation in which $y_i^{(\theta)}(t) \leqslant X_i^{(\theta)}(t) \leqslant Y_i^{(\theta)}(t)$ and $y_i^{(\theta)}(t) \leqslant 0 \leqslant Y_i^{(\theta)}(t)$ for all $t \geqslant t_0$, and $Y_i^{(\theta)}(\infty) > y_i^{(\theta)}(\infty)$. We need only then apply a straightforward variant of Lemma 10 to show that $X_i^{(\theta)}(t)$ must lie too close to both $Y_i^{(\theta)}(\infty)$ and $y_i^{(\theta)}(\infty)$ to permit these limits to differ. Theorem 2 is hereby completely proved.

COROLLARY 1 (Stability is graded in τ). *If $\alpha > \beta$ and (36) holds, then Theorem 2 is true for all $n \geqslant 2$ and all $\tau \geqslant \tau_0$ if $\sigma_1(\tau_0) > 0$. In particular, if $u > \alpha - \beta > 0$ and (36) holds, then Theorem 2 is true for all $n \geqslant 2$ and $\tau \geqslant 0$.*

PROOF. $\alpha > \beta$ implies $s(\tau) < 0$ for all $\tau \geqslant 0$.

COROLLARY 2. *If $\alpha > \beta$, $\sigma(\tau_0) > 0$, and (36) is true, then both Theorem and Theorem 2 hold for all $n \geqslant 2$ and $\tau \geqslant \tau_0$.*

PROOF. $\sigma_1(\tau_0) > \sigma(\tau_0)$ if $\alpha > \beta$.

10. LEARNING THEORETICAL REMARKS ON THEOREM 2

(a) *Practice Makes Perfect*

Theorem 2 describes a learning experiment performed on a machine M in which the experimenter E tries to teach M the probability distribution $\{\theta_j : j = 1, 2,\ldots, n\}$ by perturbing M with inputs $I_j(t) = \theta_j I(t)$. This experiment takes infinitely long to carry out since $I(t)$ is positive for arbitrarily large values of t. We denote such an experiment by the symbol $G^{(\infty)}$. Since no realistic experiment takes infinitely long to perform, we replace $G^{(\infty)}$ by a sequence $G^{(1)}, G^{(2)},..., G^{(N)},...$ of finite experiments which are "truncations of $G^{(\infty)}$. That is, (1) the inputs of $G^{(N)}$ are

$$I_j^{(N)} = \theta_j I(t)\, \chi(t - U(N)),$$

$j = 1, 2,..., n$, where $U(N)$ is a monotone increasing and positive function of $N \geq 1$ such that $\lim_{N\to\infty} U(N) = \infty$; and (2) the initial data of $G^{(N)}$ is the same as that of $G^{(\infty)}$. Denoting the functions of $G^{(N)}$ by superscripts "(N)" (e.g., y_{ij} is $y_{ij}^{(N)}$), we immediately find the following corollary of Theorems 1 and 2.

Corollary 3. *Given any sequence $G^{(1)}, G^{(2)},..., G^{(N)}$ of truncations of a $G^{(\infty)}$ such that $\alpha > \beta$, $\sigma(\tau) > 0$, and*

$$\int e^{-\alpha(t-v)} I(v)\, dv \geq k, \qquad t \geq T_0$$

then

(1) *for every $N \geq 1$, the limits $Q_i^{(N)} = \lim_{t\to\infty} X_i^{(N)}(t)$ and $P_{ki}^{(N)} = \lim_{t\to\infty} y_{ki}^{(N)}(t)$ exist and are equal,*

(2) *for every $N \geq 1$ and $t \geq U(N)$, the functions $X_i(t)$ and $y_{ki}(t)$ lie in the interval $[m_i^{(N)}, M_i^{(N)}]$, where*

$$m_i^{(N)} = \min\{X_i^{(N)}(U(N)), y_i^{(N)}(U(N))\},$$

$$M_i^{(N)} = \max\{X_i^{(N)}(U(N)), Y_i^{(N)}(U(N))\},$$

and

$$\lim_{N\to\infty} m_i^{(N)} \quad \lim_{N\to\infty} M_i^{(N)} = \theta_i\,,$$

$i, k = 1, 2,..., n$. In particular,

$$\lim_{N\to\infty} Q_i^{(N)} = \lim_{N\to\infty} P_{ki}^{(N)} = \theta_i\,,$$

$i, k = 1, 2,..., n$.

(3) *for every $N \geq 1$ and $t \geq 0$, the functions $\dot{Y}_{i,\theta}^{(N)}$, $\dot{y}_{i,\theta}^{(N)}$, $X_i^{(N)} - Y_{i,\theta}^{(N)}$, and $X_i^{(N)} - y_{i,\theta}^{(N)}$ change sign at most once and not at all if $y_i^{(N)}(0) \leq X_i^{(N)}(0) \leq Y_i^{(N)}(0)$. The θ's can be erased for $t \geq U(N)$.*

Corollary 3 says that as E increases the "practice time" $U(N)$ for learning the probabilities θ_i, he can guarantee that the maximal deviation $M_i^{(N)} - m_i^{(N)}$ of $X_i^{(N)}(t)$ and $y_{ki}^{(N)}(t)$ from θ_i when practice ends at $t = U(N)$ can be made as small as he pleases. That is, "practice makes perfect."

(b) *An Isolated Machine Does Not Forget*

If nothing more is taught in $(U(N), \infty)$, then M "remembers" the probabilities θ_i with at least the same accuracy $M_i^{(N)} - m_i^{(N)}$ at all future times.

(c) *The Memory of an Isolated Machine Spontaneously Improves after Sufficient Practice*

Corollary 4 (Crispening). *For N sufficiently large and $t \geq U(N)$, one of the following cases holds:*

(1) *$Y_i^{(N)}(t) \geq X_i^{(N)}(t) \geq \theta_i$ and $Y_i^{(N)}(t)$ is monotone decreasing;*

(2) *$\theta_i \geq X_i^{(N)}(t) \geq y_i^{(N)}(t)$ and $y_i^{(N)}(t)$ is monotone increasing,*

(3) *$Y_i^{(N)}(t) \geq \theta_i \geq y_i^{(N)}(t)$, $X_i^{(N)}(t) \in [y_i^{(N)}(t), Y_i^{(N)}(t)]$, $Y_i^{(\theta)}(t)$ is monotone decreasing, and $y_i^{(\theta)}(t)$ is monotone increasing.*

Proof. These cases correspond to Cases 1–3 of Lemma 7. For example, consider the case $\theta_i = 0$. Then only (1) is possible and we show that it arises as follows.

If $X_i^{(N)}(0) > Y_i^{(N)}(0)$ then $X_i^{(N)}(t)$ decreases and $Y_i^{(N)}(t)$ increases for $t \in [0, U(N)]$ until the first $t = t_1^{(N)}$ at which $X_i^{(N)}(t) = Y_i^{(N)}(t)$. Such a t_1 must exist for all sufficiently large N, or else $X_i^{(N)}(t) \geq Y_i^{(N)}(0) > 0$ and $\lim_{N\to\infty} Q_i^{(N)} \neq 0$. For such values of N, $Y_i^{(N)}(t) \geq X_i^{(N)}(t)$ for $t \geq t_1^{(N)}$, and thus $Y_i^{(N)}$ is monotone decreasing.

Corollary 4 shows that after a vertex v_i has received enough practice, the maximal deviations $Y_i^{(N)}$, or $y_i^{(N)}$, or both, from the intended value θ_i can only *decrease* after practice ceases. This "crispening" or "spontaneous improvement" effect also occurs in outstars [1].

(d) *An Isolated Machine Remembers without Overtly Practicing*

The condition $\alpha > \beta$ which we need in learning experiments (Theorem 2) is equivalent to the assumption that all outputs $x_i^{(N)}(t)$ approach zero exponentially in $(U(N), \infty)$ as $t \to \infty$ for all $\tau \geq 0$. Or speaking heuristically, it describes the case for which outputs are produced only in response to inputs. Since for times $t \gg U(N)$, the outputs from M are negligible, E has no evidence available that M remembers the weights θ_i. It is plausible to suppose that as the effect of inputs on outputs wears off, M forgets the information contained in these outputs. This is, however, false, as Remark 10b illustrates. Thus M remembers without "overtly" practicing.

(e) *The Machine Forgets Its Past as It is Called upon to Reproduce It*

To test at times $t \gg U(N)$ whether M does indeed remember his probabilities θ_i, E perturbs a vertex v_k and observes whether θ_i of the output thereby produced comes from v_i. The following corollary shows that E cannot produce the fraction θ_i from v_i on a long sequence of "recall" experiments of this kind without substantially destroying the memory of θ_i in M, unless $\theta_i = \delta_{ik}$.

COROLLARY 5 (recall experiments). *Let* $\alpha > \beta$, $\sigma(\tau) > 0$, *and*

$$I_i(t) = \theta_i I^{(1)}(t)\,\chi(t-T) + \delta_{ik} I^{(2)}(t)\,\chi(T-t),$$

where

$$\int_T^t e^{-\alpha(t-v)} I^{(2)}(v)\,dv \geqslant k, \qquad t \geqslant T + T_0,$$

and $T \geqslant 0$. *Then* $Q_i = P_{ki} = \delta_{ik}$.

Thus, no matter what M learns in $[0, T]$ and no matter how large T is, if only v_k is perturbed in frequent recall experiments in (T, ∞), then M will eventually forget all prior learning in place of the new probabilities δ_{ik}. Whenever $\theta_i \neq \delta_{ik}$ for some fixed k, retraining experiments must be interspersed among recall experiments or all memory of prior learning will eventually be washed away. By contrast, the memory of an outstar is not damaged by recall experiments [1].

(f) *All Errors Can Be Corrected*

The previous remark is a special case of the fact that a machine trained on one set of probabilities $\theta_i^{(1)}$ for a finite amount of time can always be retrained on an arbitrary new set of probabilities $\theta_i^{(2)}$. This is because Theorems 1 and 2 hold for *all* nonnegative initial data; i.e., because our limit theorems hold *globally*.

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