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On the Variational Systems of Some Nonlinear Difference-Differential Equations

STEPHEN GROSSBERG†

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts, 02139

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1. INTRODUCTION

This paper studies the variational systems of two closely related systems of nonlinear difference-differential equations which arise in prediction- and learning-theoretical applications ([*1*], [*2*], [*3*]). The first system is

$$\dot{x}_i(t) = -\alpha x_i(t) + \beta \sum_{m=1}^{n} x_m(t-\tau)\, y_{mi}(t), \qquad (1)$$

$$y_{jk}(t) = z_{jk}(t)\left[\sum_{m=1}^{n} z_{jm}(t)\right]^{-1}, \qquad (2) \quad (*)$$

and

$$\dot{z}_{jk}(t) = -u z_{jk}(t) + \beta x_j(t-\tau)\, x_k(t), \qquad (3)$$

$i, j, k = 1, 2, \ldots, n$, where n is any integer greater than 1, τ is any nonnegative time lag, and $\beta > 0$. The second system differs from the first only in that (3) is replaced by

$$\dot{z}_{jk}(t) = -u z_{jk}(t) + \beta x_j(t-\tau)\, x_k(t), \; j \neq k \qquad (3')$$

and $\qquad (**)$

$$z_{jj}(t) \equiv 0, \qquad (3'')$$

$j, k = 1, 2, \ldots, n$. Nonetheless, the qualitative behavior of (*) and (**) differ dramatically as $t \to \infty$.

(*) and (**) can be interpreted as cross-correlated flows over directed probabilistic networks $G^{(*)}$ and $G^{(**)}$, respectively [*1*]. Both $G^{(*)}$ and $G^{(**)}$ have n vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and n^2 directed edges

$$E = \{e_{jk} : j, k = 1, 2, \ldots, n\}.$$

In $G^{(*)}$, each edge e_{jk} is assigned the weight $\varphi(e_{jk}) = 1/n$. By contrast, in $G^{(**)}$ each edge e_{jk} leading from a given vertex v_j to a *distinct* vertex v_k, $k \neq j$,

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is assigned the weight $\varphi(e_{jk}) = 1/(n-1)$, whereas each e_{jj} is assigned the weight $\varphi(e_{jj}) = 0$. Every vertex is connected with positive weight to every other vertex in both $G^{(*)}$ and $G^{(**)}$, and thus both of these graphs are *complete*. Since in $G^{(*)}$ even $\varphi(e_{jj}) > 0$, we call $G^{(*)}$ a complete graph *with loops*. Since in $G^{(**)}$, $\varphi(e_{jj}) = 0$, $G^{(*)}$ is a complete graph *without loops*. Often in the theory of nonlinear networks, the addition of loops complicates the analysis by creating a new source of nonlinear oscillations. In the present account, the reverse is true, since $G^{(*)}$ is much easier to analyse than is $G^{(**)}$.

Our reason for studying the variational systems of (*) and (**) is twofold. Actually, the global behavior of (*) itself has already been analysed [*3*]. The result is stated in terms of the ratios $y_{jk}(t)$ and the corresponding ratios $X_k(t) = x_k(t)[\sum_{m=1}^{n} x_m(t)]^{-1}$, as well as the constant $\sigma(\tau) \equiv u + 2s(\tau)$, where $s(\tau)$ is the largest real part of the zeros of $R_\tau(s) \equiv s + \alpha - \beta e^{-\tau s}$. We state some of the facts concerning the limits of y_{jk} and X_k as $t \to \infty$ below.

THEOREM 1. *For any $n \geqslant 2$ and any $\tau \geqslant 0$ with $\sigma(\tau) > 0$, let (*) have arbitrary nonnegative and continuous initial data. Then the limits*

$$Q_i = \lim_{t\to\infty} X_i(t) \qquad \text{and} \qquad P_{jk} = \lim_{t\to\infty} y_{jk}(t)$$

exist and satisfy the equations

$$P_{ji} = Q_i\,, \; i, j = 1, 2, \ldots, n.$$

Moreover

$$Q_i \in [m_i\,, M_i],$$

where

$$m_i = \min\{X_i(0), y_{ki}(0) : k = 1, 2, \ldots, n\}$$

and

$$M_i = \max\{X_i(0), y_{ki}(0) : k = 1, 2, \ldots, n\}.$$

COROLLARY 1. (*Stability is Graded in τ*). *If $\alpha > \beta$ and $\sigma(\tau_0) > 0$, then Theorem 1 holds for all $n \geqslant 2$ and $\tau \geqslant \tau_0$.*

COROLLARY 2. *If $u > 2(\alpha - \beta) > 0$, then Theorem 1 holds for all $n \geqslant 2$ and $\tau \geqslant 0$.*

Genuinely nonlinear systems of difference-differential equations which can be subjected to a global analysis are hard to find, and so we take advantage of this one to compare its global nonlinear and linearized behavior.

(**) is a much harder system to analyse. Thusfar the only global result available discusses the case $n = 3$ and $\tau = 0$ for initial data constrained by the condition $z_{ij}(0) = z_{ji}(0)$, $i, j = 1, 2, 3$[*2*]. Some of the pertinent facts in this case are listed below.

THEOREM 2. *Let (**) be given with $n = 3$, $\tau = 0$, and arbitrary positive and continuous initial data satisfying the constraint $z_{ij}(0) = z_{ji}(0)$ $i, j = 1, 2, 3$. Then the limits $Q_i = \lim_{t\to\infty} X_i(t)$ and $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ exist and satisfy the equations*

$$\tfrac{1}{2} \geqslant Q_i = Q_j P_{ji} + Q_k P_{ki}, \{i, j, k\} = \{1, 2, 3\}.$$

In particular,

$$\lim_{t\to\infty} x_i(t)\, e^{(\alpha-\beta)t} = Q_i \sum_{m=1}^{3} x_m(0).$$

If $\sigma \equiv u + 2(\beta - \alpha) > 0$, then $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1 - \delta_{jk})$. If $\sigma < 0$, then

$$| P_{jk} - y_{jk}(0)| \leqslant 2 \log \left(1 + \frac{\sum_{m=1}^{3} z_{jm}(0)}{\beta(\sum_{m=1}^{3} x_m(0))^2 \,|\, \sigma \,|}\right)$$

Since $\sigma = \sigma(0)$, we note that (**) has the unique limits $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1 - \delta_{jk})$ if $\sigma(0) > 0$. By contrast, (*) can have any limits satisfying $Q_i = P_{ji} \in [m_i, M_i]$ when $\sigma(0) > 0$.

We will be able to analyse the variational system of (**) for a large set of n and τ that includes all $n \geqslant 3$ when $\tau = 0$. This analysis will reveal a formal reason why (*) is easier to analyse than (**). Indeed, the behavior of (*)'s variational system depends on the behavior of a linear second-order differential equation of the form

$$\ddot{g} + A(t)\,\dot{g} + B(t) g = 0,$$

whose coefficients $A(t)$ and $B(t)$ converge to constants at an exponential rate as $t \to \infty$. The behavior of (**)'s variational system, on the other hand, depends on the behavior of a linear second-order *difference*-differential equation of the form

$$\ddot{h} + \bar{A}(t)\,\dot{h} + \bar{B}(t)\,\dot{h}(t - \tau) + \bar{C}(t)\,h + \bar{D}(t)\,h(t - \tau) = 0,$$

whose coefficients $\bar{A}(t)$, $\bar{B}(t)$, $\bar{C}(t)$, and $\bar{D}(t)$ also converge to constants at an exponential rate as $t \to \infty$. Adding loops to the linearized graph thus transforms the problem from one in difference-differential equations to a simpler one in differential equations. This fact also shows that we should expect (*) and (**) to be affected in different ways by an increase in the time lag τ.

Theorem 1 for (*) shows that the limiting values Q_i can occur anywhere in the interval $[m_i, M_i]$ and are not, in particular, fixed numbers independent of initial data, as we find in Theorem 2 when $\sigma(0) > 0$. This fact is translated in the variational system of (*) as follows: the coefficient $B(t)$ converges exponentially to 0 as $t \to \infty$. Thus the equation for $g(t)$ reduces as $t \to \infty$ to an equation in which only terms involving derivatives of g are important, and an extra degree of freedom in determining g's limiting behavior as $t \to \infty$ is acquired. This is not true of the equation for $h(t)$ as $t \to \infty$, and thus h's behavior is determined without an extra degree of freedom as $t \to \infty$.

2. THE VARIATIONAL SYSTEM OF (*)

We now linearize (*) and compare the behavior of the linearized system with (*) itself. Speaking roughly, the conditions guarenteeing stability in the linearized and nonlinear versions of (*) are the same, but the distributions of values of the unknown variables as $t \to \infty$ differ. The distribution $P_{ki} = Q_i = \theta_i$ of Theorem 1 is needed to carry out our prediction theory [3], since it represents a graph which has "learned" the probability distribution $\{\theta_i : = 1, 2,..., n\}$ and can reproduce this distribution on demand. The linearized distribution can agree with this nonlinear distribution only if $P_{ki} = Q_i = 1/n$, which represents a situation of "maximal ignorance"; that is, one in which no "learning" has occurred. Thus passing from the nonlinear to the linearized case obliterates the main property for which the nonlinear system was constructed.

We now briefly review the way in which (*) is linearized. For convenience, we write (*) in matrix form as

$$\dot{U}(t) = f(U(t), U(t - \tau)) \qquad (*)$$

in terms of the $n(n + 1)$ dimensional vectors

$$U = (x_1, ..., x_n, z_{11}, ..., z_{1n}, z_{21}, ..., z_{nn})$$

and

$$f = (f_1, ..., f_n, f_{11}, ..., f_{1n}, f_{21}, ..., f_{nn}),$$

where

$$f_i = -\alpha x_i + \beta \sum_{k=1}^{n} x_k(t - \tau)\, z_{ki} \left(\sum_{m=1}^{n} z_{km}\right)^{-1}$$

and

$$f_{jk} = -u z_{jk} + \beta x_j(t - \tau)\, x_k.$$

To linearize (*), we will compare an arbitrary positive solution U of (*) with a suitably chosen positive solution U_0 of (*) by studying the function $V = U - U_0$. We expand

$$\begin{aligned}\dot{V}(t) = f((U_0(t) + V(t), U_0(t - \tau) + V(t - \tau))\\ -f(U_0(t), U_0(t - \tau))\end{aligned}$$

in a Taylor's expansion ([4], p. 341) to find the equation

$$\begin{aligned}\dot{V}(t) = f_\xi(U_0(t), U_0(t - \tau))\, V(t)\\ + f_\eta(U_0(t), U_0(t - \tau))\, V(t - \tau) + o(\| V \|),\end{aligned} \qquad (4)$$

where we have written $f = f(\xi, \eta)$ as a function of two $n(n+1)$-dimensional vectors ξ and η. The closely related *linear* system

$$\dot{W}(t) = f_\xi(U_0(t), U_0(t-\tau))\, W(t) + f_\eta(U_0(t), U_0(t-\tau))\, W(t-\tau) \tag{5}$$

in which the higher-order terms $o(\| V \|)$ are entirely ignored is called the *variational system* of (*). In the following, our notation for W in component form will always be

$$W = (h_1, \ldots, h_n, h_{11}, \ldots, h_{nn}),$$

where h_i in (5) corresponds to x_i in (*), and h_{jk} corresponds to z_{jk}.

The comparison function U_0 will always be chosen in the form

$$U_0 = (\underbrace{\gamma, \ldots, \gamma}_{n \text{ times}}, \underbrace{\delta, \ldots, \delta}_{n^2 \text{ times}})$$

where γ is a positive solution of

$$\dot{\gamma}(t) = -\alpha\gamma(t) + \beta\gamma(t-\tau) \tag{6}$$

with continuously differentiable initial data in $[0, \tau]$, and δ is a positive solution of

$$\dot{\delta}(t) = -u\delta(t) + \beta\gamma(t-\tau)\,\gamma(t), \tag{7}$$

also with continuously differentiable initial data in $[0, \tau]$. It is easily seen that (6) and (7) are the solutions x_i and z_{jk} of (1) and (3) when the initial data of (*) has the form $x_i(v) = \gamma(v)$, $v \in [0, \tau]$, and $z_{jk}(\tau) = \delta(\tau)$. Any solution U_0 of this form will therefore be called a *positive uniform* solution of (*). Our desire to compare an arbitrary positive solution of (*) with a properly chosen positive uniform solution of (*) is motivated by the following proposition.

PROPOSITION 1. *Given any positive solution*

$$U = (x_1, \ldots, x_n, z_{11}, \ldots, z_{nn})$$

of (*), *there exists a positive uniform solution*

$$U_0 = (\gamma, \ldots, \gamma, \delta, \ldots, \delta)$$

of (*) *such that the difference*

$$V = U - U_0 \equiv (v_1, \ldots, v_n, v_{11}, \ldots, v_{nn})$$

satisfies the equation

$$\frac{v_i}{\sum_{k=1}^n v_k} - \frac{1}{n} = \frac{1}{1-n}\left(\frac{x_i}{\sum_{k=1}^n x_k} - \frac{1}{n}\right).$$

Thus if we knew the distribution of the ratios $v_i[\sum_{k=1}^n v_k]^{-1}$, we would automatically know the distribution of the probabilities $X_i = x_i[\sum_{k=1}^n x_k]^{-1}$.

Proof. Given fixed initial data of U, let the initial data of U_0 be chosen so that $\gamma(v) = \sum_{k=1}^n x_k(v)$, $v \in [0, \tau]$. Since both γ and $x = \sum_{k=1}^n x_k$ are solutions of (6), then $\gamma \equiv x$.

Thus

$$\begin{aligned}\frac{v_i}{\sum_{k=1}^n v_k} - \frac{1}{n} &= \frac{x_i - \gamma}{\sum_{k=1}^n x_k - n\gamma} - \frac{1}{n} \\ &= \frac{x_i - \gamma}{(1-n)\,\gamma} - \frac{1}{n} \\ &= \frac{1}{1-n}\left(\frac{x_i}{\sum_{k=1}^n x_k} - 1\right) - \frac{1}{n} \\ &= \frac{1}{1-n}\left(\frac{x_i}{\sum_{k=1}^n x_k} - \frac{1}{n}\right).\end{aligned}$$

We can now state our main theorem concerning the variational system (5) of (*). To do this, we define the functional

$$K_\tau(f) = f(\tau) + \beta\int_0^\tau f(v)\, e^{-vs(\tau)}\, dv$$

for every $\tau \geqslant 0$ and every $f_0 \in C[0, \tau]$.

THEOREM 3. *Suppose the variational system of* (*) *is given with any* $n \geqslant 2$, *any* $\tau \geqslant 0$ *such that* $\sigma(\tau) > 0$, *and any positive uniform solution* U_0 *of* (*). *Then for arbitrary continuously differentiable initial data in* $[0, \tau]$ *satisfying*

$$K_\tau\left(\sum_{m=1}^n h_m\right) \neq 0,$$

there exist constants $\tilde{Q}_i$, $i = 1, 2, \ldots, n$, *such that*

$$h_i(t)\left[\sum_{m=1}^n h_m(t)\right]^{-1} = \tilde{Q}_i + O(e^{-(k(\tau)+\sigma(\tau))t})$$

and

$$h_{jk}(t)\left[\sum_{m=1}^n h_{jm}(t)\right]^{-1} = \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j} + O(e^{-\sigma(\tau)t}),$$

$i, j, k = 1, 2, \ldots, n$, *where* $k(\tau) = \beta e^{-\tau s(\tau)}$.

Moreover

$$h_i(t)\, e^{-s(\tau)t} = \frac{\tilde{Q}_i e^{-\tau s(\tau)} K_\tau(\sum_{m=1}^n h_m)}{1 + \beta\tau e^{-\tau s(\tau)}} + O(\exp[-\min(p, k(\tau) + \sigma(\tau))t])$$

and

$$h_{jk}(t)\, e^{-2s(\tau)t} = \frac{(\tilde{Q}_j + \tilde{Q}_k)\, e^{-3\tau s(\tau)}\, K_\tau(\gamma)\, K_\tau(\sum_{m=1}^n h_m)}{(1 + \beta\tau e^{-\tau s(\tau)})^2} + O(\exp[-\min(p, k(\tau) + \sigma(\tau))\, t]),$$

$j, k = 1, 2,..., n$, *where* $p = \mathrm{Re}(s_1) - \mathrm{Re}(s_2)$, *and* $\mathrm{Re}(s_i)$ *is the* ith *largest real part of the zeros* s_i *of* $R_\tau(s) = s + \alpha - \beta e^{-\tau s}$. *In particular,*

$$h_{jk}(t)\, e^{-2s(\tau)t} = \frac{e^{-2s(\tau)}\, K_\tau(\gamma)}{1 + \beta\tau e^{-\tau s(\tau)}} (h_j(t) + h_k(t)\, e^{-s(\tau)t}) + O(\exp[-\min(p, k(\tau) + \sigma(\tau))\, t]),$$

$j, k = 1, 2,..., n$.

COROLLARY 3. *Theorem* 3 *holds for all* $n \geqslant 2$ *and* $\tau \geqslant \tau_0$ *if* $\alpha > \beta$ *and* $\sigma(\tau_0) > 0$. *If* $u > 2(\alpha - \beta) > 0$, *then Theorem* 3 *holds for all* $n \geqslant 2$ *and* $\tau \geqslant 0$, *and moreover* $s(\tau) < 0$.

Proof. The proof is same as for Corollaries 1 and 2. One most show that $\alpha > \beta$ implies $\sigma(\tau)$ is monotone increasing in $\tau \geqslant 0$, and that

$$\sigma(0) = u + 2(\beta - \alpha).$$

Remarks. (a) Since continuous initial data in $[-\tau, 0]$ gives rise to a continuously differentiable solution in $(0, \infty)$, the restriction to continuously differentiable initial data in $[0, \tau]$ is merely for the sake of convenience of exposition.

(b) The study of (5) was motivated by considering for each U a U_0 with $\gamma(\xi) = \sum_{m=1}^n x_m(\xi)$, $\xi \in [0, \tau]$, which in terms of $v_i = x_i - \gamma$ yields

$$\sum_{m=1}^n v_m(\xi) = (1 - n)\, \gamma(\xi), \qquad \xi \in [0, \tau].$$

The linearized analog of this constraint is

$$\sum_{m=1}^n h_m(\xi) = (1 - n)\, \gamma(\xi), \qquad \xi \in [0, \tau].$$

This case is covered by Theorem 3 since then

$$K_\tau\left(\sum_{m=1}^n h_m\right) = (1 - n)\, K_\tau(\gamma) < 0.$$

(c) The analog of P_{jk} in Theorem 1 is clearly

$$\tilde{P}_{jk} = \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j}.$$

The equations $P_{jk} = Q_k$ of Theorem 1 agree with these equations supposing

$$\tilde{Q}_j > 0 \quad \text{iff} \quad \tilde{P}_{jk} = \tilde{Q}_k = \frac{1}{n},$$

since if

$$\tilde{Q}_k = \tilde{P}_{jk} = \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j},$$

then

$$\tilde{Q}_k + n\tilde{Q}_j\tilde{Q}_k = \tilde{Q}_j + \tilde{Q}_k,$$

and

$$\tilde{Q}_k = \frac{1}{n}.$$

Proof of Theorem 3. The proof is divided into five steps. Step (I) consists merely in writing out the variational system in terms of its components h_i and h_{jk}. These components obey the equations

$$\dot{h}_i = -\alpha h_i + \frac{\beta}{n} h(t - \tau) + \frac{\beta\gamma(t - \tau)}{n^2\delta(t)}\left[(n - 1)\, H_i - \sum_{k \neq i} H_k\right], \tag{10}$$

and

$$\dot{h}_{jk} = -uh_{jk} + \beta[\gamma(t - \tau)h_k + \gamma h_j(t - \tau)], \tag{11}$$

where

$$h = \sum_{m=1}^n h_m \quad \text{and} \quad H_j = \sum_{m=1}^n h_{mj}.$$

Step (II) shows that the sum $h = \sum_{m=1}^n h_m$ of the solution of (10) obeys the equation

$$\dot{h} = -\alpha h + \beta h(t - \tau), \tag{12}$$

and that the sum $H = \sum_{i,j=1}^{n} h_{ij}$ obeys the equation

$$\dot{H} = -uH + \beta n(\gamma(t-\tau)\, h + \gamma h(t-\tau)). \tag{13}$$

The sums h and H are therefore independent of the distribution of the h_i's and h_{jk}'s. This fact is crucial in the remainder of the proof.

In step (III), we use (12) to simplify (10) in the following way. Each of the n equations in (10) has a right hand side which depends on all $n(n+1)$ variables h_i and h_{jk}. We will nonetheless be able to transform the ith equation into an equation in which only one unknown function appears, namely the function

$$g_i = \frac{h_i}{\sum_{m=1}^{n} h_m} - \frac{1}{n}. \tag{14}$$

A price is paid for this simplification. g_i obeys a pair of *coupled* equations, namely

$$\dot{g}_i = -Dg_i + EG_i \tag{15}$$

and

$$\dot{G}_i = -uG_i + Fg_i \tag{16}$$

where

$$D = \frac{\beta h(t-\tau)}{h}, \qquad E = \frac{\beta\gamma(t-\tau)}{n^2\delta h}, \qquad F = \beta n^2\gamma(t-\tau)\, h,$$

and

$$G_i = (n-1)\, H_i - \sum_{k\neq i} H_k.$$

(15) and (16) can be thought of as an "uncoupling" of the variables h_i from the variables h_{jk} in (10). The remarkable fact about (15) and (16) is that all terms in which the time lag τ appears are relegated to the *coefficients* D, E, and F. This fact corresponds to the following fact for (*): X_i and y_{jk} obey equations of the form

$$\dot{X}_i = \sum_{m=1}^{n} A_m(y_{mi} - X_i)$$

and

$$\dot{y}_{jk} = B_j(X_k - y_{jk}),$$

where all expressions involving X_i's evaluated at past times occur in the A_m's and B_j's [3].

In step (IV), we differentiate (15) and manipulate (15) and (16) algebraically to find a second-order equation for g_i of the form

$$\ddot{g}_i + A(t)\,\dot{g}_i + B(t)\, g_i = 0. \tag{17}$$

Such an equation for general variable coefficients $A(t)$ and $B(t)$ would provide us with little information about $g_i(t)$ as $t \to \infty$. It is fortunate that in the present situation these coefficients have limits as $t \to \infty$ and approach them at an exponential rate. Moreover, $\lim_{t\to\infty} B(t) = 0$. We can therefore compare the behavior of (17) for large t with the behavior of the solution w_i of the following trivial equation with *constant* coefficients.

$$\ddot{w}_i + A\dot{w}_i = 0 \tag{18}$$

Using this comparison, we readily prove the existence of all limits $\lim_{t\to\infty} g_i(t)$, and thus of all limits $\tilde{Q}_i = \lim_{t\to\infty} h_i(t)[\sum_{m=1} h_m(t)]^{-1}$. Moreover these limits are approached at the exponential rate A.

We cannot expect the limit $\tilde{Q}_i$ to take on the same value for all initial data of (5) because of the appearance of an arbitrary constant in the integrated form of (18) as $t \to \infty$. Thus the nonuniqueness of the limits Q_i in (*) transforms in (5) to the statement that $g_i(t)$ behaves for large t like the solution $w_i(t)$ of the second order equation (18) which has no term containing $w_i(t)$.

In step (V), we imitate the method used on (10) as far as possible on (11). In this way, we show that the ratio

$$H_{jk} = h_{jk}\left[\sum_{m=1}^{n} h_{jm}\right]^{-1}$$

obeys an equation of the form

$$\dot{H}_{jk} = A_j(G_{jk} - H_{jk}) \tag{19}$$

where

(a) $A_j = \frac{d}{dt}\log \Gamma_j$;

(b) Γ_j can be written in the form

$$\Gamma_j = e^{\sigma(\tau)t}[\mu + e^{-kt}M_j(t)]$$

for some $\mu \neq 0$, $k > 0$, and bounded M_j; and

(c) G_{jk} has the limit $\frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j}$.

These facts suffice to prove the exponential convergence of H_{jk} to $\tilde{P}_{jk}$.

(I) *The Variational System in Component Form*

The computation leading to (10) and (11) is straightforward but tedious. Hence we merely give two examples of how the uniformity of U_0 enters into it. In terms of the $n(n+1)$-vectors

$$\xi = (\xi_1, \dots, \xi_n, \xi_{11}, \dots, \xi_{nn})$$
$$\eta = (\eta_1, \dots, \eta_n, \eta_{11}, \dots, \eta_{nn}),$$

the vector $f = f(\xi, \eta) \equiv (f_1, \dots, f_n, f_{11}, \dots, f_{nn})$ has components

$$f_i = f_i(\xi, n) = -\alpha\xi_i + \beta \sum_{k=1}^{n} \eta_k \xi_{ki} \left(\sum_{m=1}^{n} \xi_{km}\right)^{-1}$$

and

$$f_{jk} = f_{jk}(\xi, \eta) = -u\xi_{jk} + \beta\eta_j\xi_k .$$

Clearly

(1) $$\frac{\partial f_i}{\partial \xi_{ji}} = \frac{\beta\eta_j}{(\sum_{m=1}^n \xi_{jm})^2}\left[\sum_{m=1}^{n} \xi_{jm} - \xi_{ji}\right],$$

so that by the uniformity of U_0,

$$\frac{\partial f_i}{\partial \xi_{ji}}(U_0(t), U_0(t-\tau)) = \frac{\beta(n-1)\,\gamma(t-\tau)}{n^2\delta(t)},$$

which is independent of i and j; and

(2) $$\frac{\partial f_i}{\partial \eta_j} = \frac{\beta\xi_{ji}}{\sum_{m=1}^n \xi_{jm}},$$

so that

$$\frac{\partial f_i}{\partial \eta_j}(U_0(t), U_0(t-\tau)) = \frac{\beta}{n},$$

which is constant.

(II) *Equations for h and H*

(12) is derived by summing over i in (10) and noticing that in both $(n-1)\sum_{i=1}^n H_i$ and $\sum_{i=1}^n \sum_{k\neq i} H_k$, each h_{ij} occurs exactly $n-1$ times.

(13) is an immediate consequence of summing (11) over $i, j = 1, 2, \dots, n$.

(III) *Uncoupling the Functions h_i From the Functions h_{jk}*

The derivation of (15) and (16) for $g_i = (h_i/h) - (1/n)$ depends crucially on the fact that h and H are independent of the distribution of the h_i's and h_{jk}'s.

First we derive an equation for $\lambda_i = h_i/h$. Since

$$\dot\lambda_i = \frac{1}{h}\left(\dot h_i - h_i\frac{\dot h}{h}\right),$$

we have by (10) and (12), using the notation $G_i = (n-1)H_i - \sum_{k\neq i} H_k$, that

$$\dot\lambda_i = \frac{1}{h}\left[-\alpha h_i + \frac{\beta}{n}h(t-\tau) + \frac{\beta\gamma(t-\tau)}{n^2\delta}G_i - h_i\left(-\alpha + \frac{\beta h(t-\tau)}{h}\right)\right]$$
$$= -\frac{\beta h(t-\tau)}{h}\left(\lambda_i - \frac{1}{n}\right) + \frac{\beta\gamma(t-\tau)}{n^2\delta h}G_i .$$

Letting $D = \beta h(t-\tau)/h$ and $E = \beta\gamma(t-\tau)/n^2\delta h$, we immediately find (15) since $\dot g_i = \dot\lambda_i$. (16) is derived as follows. By (11),

$$\dot G_i = (n-1)\dot H_i - \sum_{k\neq i} \dot H_k$$
$$= -uG_i + \beta(n-1)[n\gamma(t-\tau)\,h_i + \gamma h(t-\tau)]$$
$$-\beta\left[n\gamma(t-\tau)\sum_{k\neq i} h_k + (n-1)\,\gamma h(t-\tau)\right].$$

Cancelling terms, we find

$$\dot G_i = -uG_i + \beta n\gamma(t-\tau)\left[(n-1)\,h_i - \sum_{k\neq 1} h_k\right]$$
$$= -uG_i + \beta n\gamma(t-\tau)[nh_i - h]$$
$$= -uG_i + \beta n^2\gamma(t-\tau)\,hg_i$$
$$= -uG_i + Fg_i,$$

where $F = \beta n^2\gamma(t-\tau)\,h$ is independent of the distribution of h_i's.

(IV) *A Second Order Equation for g_i*

Differentiating (15), we find

$$\ddot g_i = -\dot D g_i - D\dot g_i + \dot E G_i + E\dot G_i . \tag{20}$$

Substituting (16) into (20) yields

$$\ddot g_i = (EF - \dot D)\,g_i - D\dot g_i + (\dot E - uE)\,G_i . \tag{21}$$

G_i is eliminated from this expression through the use of (15), which when substituted into (21) gives

$$\ddot{g}_i + A(t)\dot{g}_i + B(t) g_i = 0, \tag{17}$$

where

$$A(t) = D(t) + u - \frac{\dot{E}(t)}{E(t)}$$

and

$$B(t) = D(t)\left(u - \frac{\dot{E}(t)}{E(t)}\right) + \dot{D}(t) - E(t)F(t).$$

That is,

$$A(t) = \frac{\beta h(t-\tau)}{h} + u - \frac{d}{dt}\log\frac{\gamma(t-\tau)}{h\delta}$$

and

$$B(t) = \frac{\beta h(t-\tau)}{h}\left(u - \frac{d}{dt}\log\frac{\gamma(t-\tau)}{h\delta}\right) + \beta\left(\frac{h(t-\tau)}{h}\right)^{\cdot} - \frac{\beta^2\gamma^2(t-\tau)}{\delta}.$$

In order to describe the behavior of $A(t)$ and $B(t)$ for large t, we need the following lemma.

LEMMA 1. *Let γ and δ be any solutions of the equations* (6) *and* (7), *respectively, whose initial data is continuously differentiable in* $[0, \tau]$ *and, moreover,* $K_\tau(\gamma) \neq 0$. *Then*

$$\lim_{t\to\infty}\frac{\gamma(t-\tau)}{\gamma(t)} = e^{-\tau s(\tau)}, \tag{22}$$

$$\lim_{t\to\infty}\frac{d}{dt}\frac{\gamma(t-\tau)}{\gamma(t)} = 0, \tag{23}$$

and

$$\lim_{t\to\infty}\frac{\gamma^2(t-\tau)}{\delta(t)} = \frac{1}{\beta}\sigma(\tau)\, e^{-\tau s(\tau)}. \tag{24}$$

If γ_1 and γ_2 are any two solutions of (6) *chosen in this way, and δ_1 is a solution of*

$$\dot{\delta}_1(t) = -u\delta_1(t) + \beta\gamma_1(t-\tau)\,\gamma_1(t),$$

then

$$\lim_{t\to\infty}\frac{d}{dt}\log\frac{\gamma_2(t)\,\delta_1(t)}{\gamma_1(t-\tau)} = \sigma(\tau) - u. \tag{25}$$

There also exists a $t_0 > 0$ such that

$$\int_{t_0}^{\infty}\left|\frac{\gamma(v-\tau)}{\gamma(v)} - e^{-\tau s(\tau)}\right| dv < \infty, \tag{26}$$

$$\int_{t_0}^{\infty}\left|\frac{d}{dv}\frac{\gamma(v-\tau)}{\gamma(v)}\right| dv < \infty, \tag{27}$$

$$\int_{t_0}^{\infty}\left|\frac{\gamma^2(v-\tau)}{\delta(v)} - \frac{1}{\beta}\sigma(\tau)\, e^{-\tau s(\tau)}\right| dv < \infty, \tag{28}$$

and

$$\int_{t_0}^{\infty}\left|\frac{d}{dv}\log\frac{\gamma_2(v)\,\delta_1(v)}{\gamma_1(v-\tau)} + u - \sigma(\tau)\right| dv < \infty. \tag{29}$$

Proof. The proof is based on the standard representation of γ as an infinite series

$$\gamma(t) = \sum_{m=1}^{\infty} c_m e^{s_m t},$$

where $s_1, s_2, \ldots,$ are the zeros of $R_\tau(s) \equiv s + \alpha - \beta e^{-\tau s}$ arranged with $\mathrm{Re}(s_m) \geqslant \mathrm{Re}(s_{m+1})$, $m = 1, 2, \ldots$ ([5], p. 109). From this can be derived the closed form representation

$$\gamma(t) = e^{s(\tau)t}\left[\frac{e^{-\tau s(\tau)}K_\tau(\gamma)}{1 + \beta\tau e^{-\tau s(\tau)}} + e^{-pt}H(t)\right], \tag{30}$$

where $p = \mathrm{Re}(s_1) - \mathrm{Re}(s_2) = s(\tau) - \mathrm{Re}(s_2) > 0$, and H is bounded ([3], Lemma 3).

(22) is an immediate consequence of (30). (23) follows from

$$\frac{d}{dt}\frac{\gamma(t-\tau)}{\gamma(t)} = \frac{\gamma(t-\tau)}{\gamma(t)}\left[\frac{\dot{\gamma}(t-\tau)}{\gamma(t-\tau)} - \frac{\dot{\gamma}(t)}{\gamma(t)}\right]$$

and

$$\frac{\dot{\gamma}(t)}{\gamma(t)} = -\alpha + \frac{\beta\gamma(t-\tau)}{\gamma(t)},$$

since then

$$\frac{d}{dt}\frac{\gamma(t-\tau)}{\gamma(t)} = \frac{\beta\gamma(t-\tau)}{\gamma(t)}\left[\frac{\gamma(t-2\tau)}{\gamma(t-\tau)} - \frac{\gamma(t-\tau)}{\gamma(t)}\right],$$

and the result follows by invoking (22).

(24) follows from

$$\delta(t) = e^{-ut}\left[\delta(\tau)\, e^{u\tau} + \beta\int_\tau^t e^{uv}\gamma(v-\tau)\,\gamma(v)\, dv\right], \qquad t \geqslant \tau,$$

where

$$\gamma(v-\tau)\,\gamma(v) = e^{2s(\tau)v}e^{-\tau s(\tau)}[c_1^2 + e^{-pv}R(v)]$$

and R is bounded. Thus

$$\delta(t) = e^{-ut}\left[\delta(\tau)\,e^{u\tau} + \beta c_1^2 e^{-\tau s(\tau)}\left(\frac{e^{\sigma(\tau)t} - e^{\sigma(\tau)\tau}}{\sigma(\tau)}\right) + \beta e^{-\tau s(\tau)}\int_\tau^t e^{(\sigma(\tau)-p)v}R(v)\,dv\right]$$

and

$$\delta(t) = e^{-ut}\left[\frac{\beta c_1^2 e^{-\tau s(\tau)}e^{\sigma(\tau)t}}{\sigma(\tau)} + k + e^{(\sigma(\tau)-p)t}R_1(t)\right],$$

where

$$k = \delta(\tau)\,e^{u\tau} - \beta c_1^2(\sigma(\tau))^{-1}\,e^{\tau(u+s(\tau))}$$

and R_1 is bounded. Along with

$$\gamma^2(t-\tau) = e^{2s(\tau)t}e^{-2\tau s(\tau)}[c_1^2 + e^{-pt}R_2(t)],$$

where R_2 is bounded, we readily find (24).

(25) follows from the identities

$$\frac{d}{dt}\log\frac{\gamma_2(t)\,\delta_1(t)}{\gamma_1(t-\tau)}$$

$$= \frac{\dot\gamma_2(t)}{\gamma_2(t)} + \frac{\dot\delta_1(t)}{\delta_1(t)} - \frac{\dot\gamma_1(t-\tau)}{\gamma_1(t-\tau)}$$

$$= \beta\left[\frac{\gamma_2(t-\tau)}{\gamma_2(t)} + \frac{\gamma_1(t-\tau)\,\gamma_1(t)}{\delta_1(t)} - \frac{\gamma_1(t-2\tau)}{\gamma_1(t-\tau)}\right] - u$$

$$= \beta\left[\frac{\gamma_2(t-\tau)}{\gamma_2(t)} + \left(\frac{\gamma_1^2(t-\tau)}{\delta_1(t)}\right)\left(\frac{\gamma_1(t)}{\gamma_1(t-\tau)}\right) - \frac{\gamma_1(t-2\tau)}{\gamma_1(t-\tau)}\right] - u,$$

along with (22) and (24).

To prove (26), note that for t sufficiently large,

$$\frac{\gamma(t-\tau)}{\gamma(t)} - e^{-\tau s(\tau)} = \frac{e^{s(\tau)(t-\tau)}(c_1 + e^{-p(t-\tau)}H(t-\tau))}{e^{s(\tau)t}(c_1 + e^{-pt}H(t))} - e^{-\tau s(\tau)}$$

$$= e^{-\tau s(\tau)}(1 + e^{-pt}O(1)) - e^{-\tau s(\tau)}$$

$$= e^{-pt}O(1).$$

(26) follows immediately. (27)–(29) can be proved in an identical way.

Lemma 1 implies that the limits $A = \lim_{t\to\infty} A(t)$ and $B = \lim_{t\to\infty} B(t)$ exist and are approached at an exponential rate as $t \to \infty$. A and B are readily found to equal $A = k(\tau) + \sigma(\tau)$ and $B = 0$, where we have set $k(\tau) = \beta e^{-\tau s(\tau)}$.

We can therefore compare the behavior for large t of g_i with the behavior of w_i in

$$\ddot w_i + A\dot w_i = 0, \tag{18}$$

by [6], Chapter 2. Since the characteristic roots of (18) are $k = 0$ and $-A$,

$$w_i(t) = C_1 + C_2 e^{-At},$$

where C_1 and C_2 are constants. Since $w_i(t)$ thus converges at the exponential rate A to C_1, it follows that all $g_i(t)$ converge at the exponential rate A to constants as $t \to \infty$, and thus there exist constants $\tilde Q_i$ such that

$$h_i(t)\left[\sum_{k=1}^n h_k(t)\right]^{-1} = \tilde Q_i + O(e^{-(k(\tau)+\sigma(\tau))t}). \tag{31}$$

The fact that

$$h_k(t)\,e^{-s(\tau)t} = \frac{\tilde Q_i e^{-\tau s(\tau)} K_\tau(\sum_{m=1}^n h_m)}{1 + \beta\tau e^{-\tau s(\tau)}} + O(\exp[-\min(p, k(\tau) + \sigma(\tau))\,t]), \tag{32}$$

$i = 1, 2,\ldots, n$, follows by substituting (30) for $\sum_{k=1}^n h_k(t)$ into (31) and rearranging terms.

It is easily seen from the above argument that the limits $\tilde Q_i$ exist for all values of A, and hence for all real choices of α, β, and u with $\beta > 0$. $A > 0$, which occurs whenever $\sigma(\tau) > 0$, merely guarantees that these limits are *finite*.

(V) *An Equation for* $H_{jk} = h_{jk}[\sum_{m=1}^n h_{jm}]^{-1}$.

Let $H^{(j)} = \sum_{m=1}^n h_{jm}$. Then $H_{jk} = h_{jk}/H^{(j)}$, and

$$\dot H_{jk} = \frac{1}{H^{(j)}}\left(\dot h_{jk} - h_{jk}\frac{\dot H^{(j)}}{H^{(j)}}\right),$$

where, by (11), $H^{(j)}$ obeys the equation

$$\dot H^{(j)} = -uH^{(j)} + \beta[\gamma(t-\tau)\,h + n\gamma h_j(t-\tau)].$$

Thus, again by (11),

$$\dot{H}_{jk} = \frac{1}{H^{(j)}}\left[-uh_{jk} + \beta(\gamma(t-\tau)\,h_k + \gamma h_j(t-\tau))\right.$$
$$\left. - h_{jk}\left(-u + \frac{\beta(\gamma(t-\tau)\,h + n\gamma h_j(t-\tau))}{H^{(j)}}\right)\right]$$
$$= \frac{\beta}{H^{(j)}}[\gamma(t-\tau)\,h_k + \gamma h_j(t-\tau) - H_{jk}(\gamma(t-\tau)\,h + n\gamma h_j(t-\tau))]$$
$$= \Lambda_j(G_{jk} - H_{jk}), \tag{19}$$

using the definitions

$$\Lambda_j = \frac{\beta[\gamma(t-\tau)\,h + n\gamma h_j(t-\tau)]}{H^{(j)}}$$

and

$$G_{jk} = \frac{\gamma(t-\tau)\,h_k + \gamma h_j(t-\tau)}{\gamma(t-\tau)\,h + n\gamma h_j(t-\tau)}. \tag{33}$$

Letting

$$\Gamma_j(t) = H^{(j)}(0) + \beta\int_0^t e^{uv}[\gamma(v-\tau)\,h + n\gamma h_j(v-\tau)]\,dv, \tag{34}$$

we can write Λ_j in the form

$$\Lambda_j = \frac{d}{dt}\log\Gamma_j. \tag{35}$$

We have defined $\Gamma_j(t)$ by integrating from 0 to t rather than τ to t to avoid carrying an extra factor of $e^{u\tau}$ in forthcoming computations, and apply all representations of γ and h in $[-\tau, \tau]$ as well as in (τ, ∞). Our conclusions will not be hereby altered, since we are concerned only with limiting behavior as $t \to \infty$.

To prove that

$$H_{jk}(t) = \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j} + O(e^{-\sigma(\tau)t}), \tag{36}$$

we first show that

$$G_{jk}(t) = \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j} + O(\exp[-(\sigma(\tau) + k(\tau))\,t]). \tag{37}$$

This we do by dividing numerator and denominator of $G_{jk}(t)$ by $\gamma(t-\tau)\,h(t)$ and then invoking the definition $\lambda_j = h_j h^{-1}$. (33) becomes

$$G_{jk} = \frac{\lambda_k + \dfrac{\gamma}{\gamma(t-\tau)}\,\dfrac{h_j(t-\tau)}{h}}{1 + \dfrac{h\gamma}{\gamma(t-\tau)}\,\dfrac{h_j(t-\tau)}{h}}$$
$$= \frac{\lambda_k + \dfrac{\gamma h(t-\tau)}{\gamma(t-\tau)h}\lambda_j(t-\tau)}{1 + n\dfrac{\gamma h(t-\tau)}{\gamma(t-\tau)h}\lambda_j(t-\tau)}.$$

By Lemma 1,

$$\lim_{t\to\infty}\frac{\gamma h(t-\tau)}{\gamma(t-\tau)h} = 1,$$

and by (31),

$$\lambda_i(t) = \tilde{Q}_i + O(\exp[-(k(\tau) + \sigma(\tau))\,t]),$$

$i = 1, 2,\ldots, n$. (37) follows immediately.

We now prove (36). Write (19) as

$$\dot{g}_{jk} = \Lambda_j(\bar{G}_{jk} - g_{jk}), \tag{38}$$

in terms of

$$g_{jk} = H_{jk} - \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j}$$

and

$$\bar{G}_{jk} = G_{jk} - \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j}.$$

Integrate (38). Then

$$g_{jk}(t) = \exp\left(-\int_0^t \Lambda_j\,dw\right)\left[g_{jk}(0) + \int_0^t \bar{G}_{jk}\Lambda_j\exp\left(\int_0^v \Lambda_j\,dw\right)\right].$$

By (35), $\exp[\int_w^t \Lambda_j\,dw] = \Gamma_j(t)\,\Gamma_j^{-1}(w)$, and so g_{jk} can be written as

$$g_{jk}(t) = A_{jk}(t) + B_{jk}(t), \tag{39}$$

if we let

$$A_{jk}(t) = g_{jk}(0)\,\Gamma_j(0)\,\Gamma_j^{-1}(t), \tag{40}$$

and

$$B_{jk}(t) = \Gamma_j^{-1}(t)\int_0^t \dot{\Gamma}_j(v)\,\bar{G}_{jk}(v)\,dv.$$

We wish to show that $g_{jk}(t) = O(e^{-\sigma(\tau)t})$. First we show that

$$A_{jk}(t) = O(e^{-\sigma(\tau)t})$$

by proving that $\Gamma_j(t) = O(e^{\sigma(\tau)t})$. By (30),

$$\gamma(t) = e^{s(\tau)t}[c_\gamma + e^{-p_\gamma t}H_\gamma(t)]$$

and

$$h(t) = e^{s(\tau)t}[c_h + e^{-p_h t}H_h(t)],$$

where $c_\gamma \neq 0 \neq c_h$, $p_\gamma > 0$, $p_h > 0$, and H_γ and H_h are bounded. (34) therefore implies

$$\Gamma_j(t) = H^{(j)}(0) + \beta e^{\sigma(\tau)t}\left[\frac{c_\gamma c_h}{\sigma(\tau)}(1 - e^{-\sigma(\tau)t}) + e^{-pt}R_j(t)\right],$$

where R_j is bounded, and so $A_{jk}(t) = O(e^{-\sigma(\tau)t})$.

Now consider B_{jk}. Since by (34)

$$\dot{\Gamma}_j(v) = \beta e^{uv}[\gamma(v-\tau)\,h + n\gamma h_j(v-\tau)],$$

we find

$$\dot{\Gamma}_j(v) = \beta e^{\sigma(\tau)v}[c_\gamma c_h + e^{-pv}M_j(v)],$$

where M_j is bounded. Thus

$$\dot{\Gamma}_j(v)\,\Gamma_j^{-1}(t) = e^{-\sigma(\tau)(t-v)}\left[\frac{c_\gamma c_h + e^{-pv}M_j(v)}{\frac{c_\gamma c_h}{\sigma(\tau)}(1-e^{-\sigma(\tau)t}) + \frac{1}{\beta}H^{(j)}(0)e^{-\sigma(\tau)t} + e^{-pt}R_j(t)}\right]$$

and given any sufficiently small $\epsilon > 0$, we can find a $T(\epsilon)$ such that $t \geqslant v \geqslant T(\epsilon)$ implies

$$|\dot{\Gamma}_j(v)\,\Gamma_j^{-1}(t)| \leqslant \exp[-\sigma(\tau)(t-v)]\left[\frac{|c_\gamma c_h| + \epsilon}{\frac{|c_\gamma c_h|}{\sigma(\tau)} - \epsilon}\right]$$

Let $\epsilon = |c_\gamma c_h|/2\sigma(\tau)$ and $T = T(c_\gamma c_h/2\sigma(\tau))$. Then B_{jk} can be broken into a sum of the two parts

$$C_{jk}(t) = \Gamma_j^{-1}(t)\int_0^T \dot{\Gamma}_j(v)\,\bar{G}_{jk}(v)\,dv$$

and

$$D_{jk}(t) = \Gamma_j^{-1}(t)\int_T^t \dot{\Gamma}_j(v)\,\bar{G}_{jk}(v)\,dv.$$

Obviously $C_{jk}(t) = O(e^{-\sigma(\tau)t})$, whereas

$$|D_{jk}(t)| \leqslant (1+2\sigma(\tau))\int_T^t \exp[-\sigma(\tau)(t-v)]\,|\bar{G}_{jk}(v)|\,dv,$$

and so by (37), $D_{jk}(t) = O(e^{-\sigma(\tau)t})$ as well. From the equation

$$H^{(j)}(t) = e^{-ut}[H^{(j)}(0) + \beta\int_0^t e^{uv}[\gamma(v-\tau)\,h + n\gamma h(v-\tau)\,\lambda_j(v-\tau)]\,dv$$

along with $g_{jk} = O(e^{-\sigma(\tau)t})$, it now readily follows that

$$h_{jk}(t)\,e^{-2s(\tau)t} = \frac{(\tilde{Q}_j + \tilde{Q}_k)\,e^{-3\tau s(\tau)}K_\tau(\sum_{m=1}^n h_m)\,K_\tau(\gamma)}{(1+\beta\tau e^{-\tau s(\tau)})^2} + O(\exp[-\min(p, k(\tau)+\sigma(\tau))\,t]),$$

for all $j, k = 1, 2,\ldots, n$, and thus by (32) that

$$h_{jk}(t)\,e^{-2s(\tau)t} = \frac{e^{-2s(\tau)}K_\tau(\gamma)}{1+\beta\tau e^{-\tau s(\tau)}}(h_j(t) + h_k(t))\,e^{-s(\tau)t} + O(\exp[-\min(p, k(\tau)+\sigma(\tau))\,t]).$$

This completes the proof of Theorem 3.

3. The Variational System of (**)

Our study of the variational system of (**) imitates the previous development as far as possible. Again we motivate our analysis by a comparison of an arbitrary positive solution U of (**) with a suitably chosen positive uniform solution U_0 of (**). Since all $z_{jj} \equiv 0$ in (**), we consider only the n^2 dimensional vector function

$$U = (x_1,\ldots, x_n, z_{12},\ldots, z_{1n},\ldots, z_{n1},\ldots, z_{n,n-1}),$$

which we call a *positive* solution if all of its entries are always positive. Similarly, a *positive uniform* solution of (**) is one of the form

$$U_0 = (\underbrace{\gamma,\ldots,\gamma}_{n\text{ times}},\ \underbrace{\delta,\ldots,\delta}_{n(n-1)\text{ times}}).$$

With these slight modifications in the definition of positive solutions in mind, we again form $V = U - U_0$ and study the variational system derived therefrom. We denote the solution of this system, which is again of the form

$$\dot{W}(t) = f_\xi(U_0(t), U_0(t-\tau))\,W(t) + f_\eta(U_0(t), U_0(t-\tau))\,W(t-\tau),$$

by $W = (h_1,\ldots, h_n, h_{12},\ldots, h_{n,n-1})$. We will prove the following theorem concerning W.

THEOREM 4. *Suppose the varational system of (**) is given with any* $n > 2$, *and any* $\tau \geqslant 0$ *such that* $\sigma(\tau) > 0$ *and*

$$k(\tau) + \sigma(\tau) > \frac{1}{n-1}k(\tau)(1+\tau\sigma(\tau)), \tag{41}$$

where $k(\tau) = \beta e^{-\tau s(\tau)}$. *Let* U_0 *be any positive uniform solution of (**). Then there exist positive constants* ω_1 *and* ω_2 *such that for any initial data satisfying* $K_\tau(\sum_{m=1}^n h_m) \neq 0$,

$$h_i(t)\left[\sum_{m=1}^n h_m(t)\right]^{-1} = \frac{1}{n} + O(e^{-\omega_1 t}) \tag{42}$$

and

$$h_{jk}(t)\left[\sum_{\substack{m=1\\m\neq j}}^{n} h_{jm}(t)\right]^{-1} = \frac{1}{n-1} + O(e^{-\omega_2 t}). \tag{43}$$

Before proving Theorem 4, we state the following corollaries to illustrate the meaning of the condition (41).

COROLLARY 4. *Theorem* 4 *holds for all* $n \geqslant 3$ *when* $\tau = 0$ *if* $\sigma(0) > 0$ *and* $\sum_{m=1}^{n} h_m(0) \neq 0$.

Proof. Then (41) becomes

$$\beta + \sigma(0) > \frac{\beta}{n-1},$$

which is automatically fulfilled if $\sigma(0) > 0$, and

$$K_0\left(\sum_{m=1}^{n} h_m\right) = \sum_{m=1}^{n} h_m(0).$$

Remark. Theorem 2 proves that for $n = 3$ and $\tau = 0$, the following limits hold if $z_{ij}(0) = z_{ji}(0)$, $i, j = 1, 2, 3$:

$$\lim_{t\to\infty} x_i(t)\left[\sum_{m=1}^{3} x_m(t)\right]^{-1} = \frac{1}{3} \tag{44}$$

and

$$\lim_{t\to\infty} z_{jk}(t)\left[\sum_{\substack{m=1\\m\neq j}}^{3} z_{jm}(t)\right]^{-1} = \frac{1}{2}. \tag{45}$$

By Proposition 1, (42) is the linearized analog of (44) for $n \geqslant 3$, and (43) is the linearized analog of (45). Thus Theorem 4 suggests that the limits of Theorem 2 hold in (**) for $\tau = 0$ and all $n \geqslant 3$ with n replacing 3 in (44) and (45). Moreover, Theorem 4 holds for arbitrary positive U, not only U's constrained by $z_{ij}(0) = z_{ji}(0)$ as in Theorem 2.

COROLLARY 5. *For every* $\tau \geqslant 0$ *with* $\sigma(\tau) > 0$, *there exists an* $n = n(\tau)$ *such that Theorem* 4 *holds for* τ *and* $n(\tau)$.

Proof. Since $\sigma(\tau) > 0$, $k(\tau) + \sigma(\tau) > 0$. Let n increase until (41) is satisfied.

COROLLARY 6 (Stability is Graded in n). *Suppose Theorem* 4 *holds for* $n = n_0$ *and* $\tau = \tau_0$. *Then it also holds for all* $n \geqslant n_0$ *and* $\tau = \tau_0$, *if* $\sigma(\tau_0) \geqslant 0$.

Proof. The right hand side of (41) is a monotone decreasing function of n.

These corollaries can be strengthened in the case $\alpha > \beta > 0$, for which our prediction theory has a sensible interpretation.

COROLLARY 7. *If* $u > 2(\alpha - \beta) > 0$, *then there exists an interval* $[0, \lambda(n)]$, *where* $\lambda(n)$ *is monotone increasing in* n *and* $\lim_{n\to\infty} \lambda(n) = \infty$, *such that Theorem* 4 *holds for* n *and all* $\tau \in [0, \lambda(n)]$.

Proof. If $\alpha > \beta > 0$, then $s(\tau) < 0$ and $s(\tau)$ is monotone increasing in $\tau \geqslant 0$ [3]. Thus $\sigma(\tau) \geqslant \sigma(0) = u + 2(\beta - \alpha) > 0$, which along with Corollaries 5 and 6 yields the desired conclusion.

4. PROOF OF THEOREM 4

The proof is carried out in six steps which imitate the proof of Theorem 3 as far as possible. We will use a similar notation to that of Theorem 3 to highlight analogies between the two theorems. Step (I) consists merely in writing out the variational system in component form. The result is

$$\dot{h}_i = -\alpha h_i + \frac{\beta\gamma(t-\tau)}{(n-1)^2\delta(t)}[(n-1)H_i - H + H^{(i)}] + \frac{\beta}{n-1} h^{(i)}(t-\tau), \tag{46}$$

where $H_i = \sum_{j\neq i} h_{ji}$, $H^{(i)} = \sum_{j\neq i} h_{ij}$, $H = \sum_{\substack{i,j\\ i\neq j}} h_{ij}$, and $h^{(i)} = \sum_{j\neq i} h_j$;

and

$$\dot{h}_{jk} = -uh_{jk} + \beta\gamma(t)\, h_j(t-\tau) + \beta\gamma(t-\tau)\, h_k. \tag{47}$$

Step (II) shows that once again the sum $h = \sum_{m=1}^{n} h_m$ obeys the linear equation

$$\dot{h}(t) = -\alpha h(t) + \beta h(t-\tau), \tag{48}$$

and that the sum H obeys the equation

$$\dot{H}(t) = -uH(t) + \beta(n-1)(\gamma(t)\, h(t-\tau) + \gamma(t-\tau)\, h(t)). \tag{49}$$

Step (III) seeks an equation for $g_i = h_i h^{-1} - 1/n$ by uncoupling the h_i's from the h_{jk}'s. The result is the following pair of coupled equations for g_i alone.

$$\dot{g}_i = Ig_i + Jg_i(t-\tau) + KG_i \tag{50}$$

and

$$\dot{G}_i = -uG_i + Lg_i \tag{51}$$

where the coefficients $I = -\beta h(t-\tau)/h$, $J = -\beta h(t-\tau)/(n-1)\,h$, $K = \beta\gamma(t-\tau)/(n-1)^2\,h\delta$, and $L = \beta n(n-2)\,\gamma(t-\tau)\,h$ are independent of the distribution of h_i's and h_{jk}'s.

Step (IV) transforms (50) and (51) into a second order equation for g_i, namely

$$\ddot{g}_i + A(t)\,\dot{g}_i + B(t)\,\dot{g}_i(t-\tau) + C(t)\,g_i + D(t)\,g_i(t-\tau) = 0. \tag{52}$$

Comparing (52) and (17), we find one basic difference between (**) and (*) in linearized form, since (52) is a *difference*-differential equation, whereas (17) is merely a differential equation. (52) is therefore considerably more difficult to analyse than is (17).

In studying (52) we are again fortunate that the limits $A = \lim_{t\to\infty} A(t)$, $B = \lim_{t\to\infty} B(t)$, $C = \lim_{t\to\infty} C(t)$, and $D = \lim_{t\to\infty} D(t)$ exist and are approached at an exponential rate. Thus we can compare $g_i(t)$ for large times t with the solution $w_i(t)$) of the following equation with constant coefficients.

$$\ddot{w}_i + A\dot{w}_i + B\dot{w}_i(t-\tau) + Cw_i + Dw_i(t-\tau) = 0. \tag{53}$$

It will actually be more useful to make a change of variables in (52) and (53) to $\xi_i = g_i e^{\lambda t}$ and $\eta_i = w_i e^{\lambda t}$ where λ is a sufficiently small positive constant. In terms of the new variables ξ_i and η_i, we find equations of the form

$$\ddot{\xi}_i + \bar{A}(t)\,\dot{\xi}_i + \bar{B}(t)\,\dot{\xi}_i(t-\tau) + \bar{C}(t)\,\xi_i + \bar{D}(t)\,\xi_i(t-\tau) = 0 \tag{54}$$

and

$$\ddot{\eta}_i + \bar{A}\dot{\eta}_i + \bar{B}\dot{\eta}_i(t-\tau) + \bar{C}\eta_i + \bar{D}\eta_i(t-\tau) = 0. \tag{55}$$

We compare equations (54) and (55) to show that ξ_i is bounded whenever η_i is bounded, and in (V) we show that η_i actually converges to zero as $t \to \infty$ because all zeros of the exponential polynomial

$$G^{(\tau)}_{\tau n\tau}(s) = s^2 + \bar{A}s + (\bar{B}s + \bar{C})\,e^{-\tau s} + \bar{D} \tag{56}$$

have negative real parts under the hypotheses of the Theorem. From these facts, it readily follows that

$$g_i \equiv h_i\left[\sum_{m=1}^{n} h_m\right]^{-1} - \frac{1}{n} = O(e^{-\lambda t}),$$

which is the first conclusion of the Theorem 4.

In step (VI), we use this fact to imitate the method used on (46) as far as possible on (47), much as we did in step (V) of Theorem 3.

(I) *The Variational System in Component Form*

The variational system is

$$\dot{W}(t) = f_\xi(U_0(t), U_0(t-\tau))\,W(t) + f_\eta(U_0(t), U_0(t-\tau))\,W(t-\tau),$$

where the vector function

$$f = f(\xi, n) = (f_1, \dots, f_n, f_{12}, \dots, f_{n,n-1})$$

of $\xi = (\xi_1, \dots, \xi_n, \xi_{12}, \dots, \xi_{n,n-1})$ and $\eta = (\eta_1, \dots, \eta_n, \eta_{12}, \dots, \eta_{n,n-1})$ is given by

$$f_i = f_i(\xi, \eta) = -\alpha\xi_i + \beta\sum_{k\neq i}\eta_k\xi_{ki}\Big(\sum_{j\neq k}\xi_{kj}\Big)^{-1}$$

and

$$f_{jk} = f_{jk}(\xi, \eta) = -u\xi_{jk} + \beta\eta_j\xi_k, \; j \neq k.$$

The computation needed to derive (46) and (47) is again straightforward but tedious, and so we merely give two examples of how the uniformity of U_0 enters it.

(1) Clearly

$$\frac{\partial f_i}{\partial \xi_{jk}} = \beta\eta_j\left[\frac{\sum_{u\neq j}\xi_{ju} - \xi_{ji}}{(\sum_{u\neq j}\xi_{ju})^2}\right] \quad \text{if} \quad j \neq i = k.$$

Thus

$$\frac{\partial f_i}{\partial \xi_{jk}}(U_0(t), U_0(t-\tau)) = \frac{\beta(n-2)\,\gamma(t-\tau)}{(n-1)^2\delta(t)},$$

which is independent of i, j, and k just so long as $j \neq i = k$.

(2) Clearly

$$\frac{\partial f_i}{\partial \eta_j} = \frac{\beta\xi_{ji}}{\sum_{k\neq j}\xi_{jk}}(1 - \delta_{ij}).$$

Thus

$$\frac{\partial f_i}{\partial \eta_j}(U_0(t), U_0(t-\tau)) = \frac{\beta}{n-1}(1 - \delta_{ij}),$$

which is constant.

(II) *Equations for the Sums* $h = \sum_{i=1}^{n} h_i$ *and* $H = \sum_{\substack{j,k \\ j \neq k}} h_{jk}$.

To see that h obeys equation (48), sum over $i = 1, 2, \ldots, n$ in (46), and apply the identities

$$(n-1)\sum_{i=1}^{n} H_i - nH + \sum_{i=1}^{n} H^{(i)} = 0$$

and

$$\sum_{i=1}^{n} h^{(i)}(t-\tau) = (n-1)\, h(t-\tau).$$

Equation (49) for H is an immediate consequence of summing over both indices j and k, $j \neq k$, in (47).

(III) *Uncoupling the Functions* h_i *from the Functions* h_{jk}.

Let $G_i = (n-1)\, H_i - H + H^{(i)}$. Then (46) becomes

$$\dot{h}_i = -\alpha h_i + \frac{\beta\gamma(t-\tau)}{(n-1)^2\delta(t)}\, G_i + \frac{\beta}{n-1}\, h^{(i)}(t-\tau). \tag{57}$$

G_i contains all the terms h_{jk} on which h_i depends. To uncouple h_i from the terms h_{jk}, we must express G_i in terms of only h_j's. In particular, we can express G_i in terms of h_i alone because, by (47) and (49),

$$\dot{G}_i = -uG_i + \beta(\tilde{H}_i - \tilde{H} + \tilde{H}^{(i)}),$$

where

$$\tilde{H}_i = (n-1)\,\gamma \sum_{j \neq i} h_j(t-\tau) + (n-1)^2\, \gamma(t-\tau)\, h_i\,,$$

$$-\tilde{H} = -(n-1)\,\gamma h(t-\tau) - (n-1)\,\gamma(t-\tau)\, h,$$

and

$$\tilde{H}^{(i)} = (n-1)\,\gamma h_i(t-\tau) + \gamma(t-\tau) \sum_{j \neq i} h_j\,.$$

Adding the first terms of these three functions gives zero. Adding the remaining terms gives

$$\begin{aligned}
\gamma(t-\tau)\Big[(n-1)^2 h_i - (n-1)\,h + \sum_{j\neq i} h_j\Big]
&= \gamma(t-\tau)[((n-1)^2 - 1)\, h_i - (n-2)\, h],\\
&= (n-2)\,\gamma(t-\tau)[nh_i - h],\\
&= (n-2)\, n\gamma(t-\tau)\, hg_i\,.
\end{aligned}$$

Thus

$$\dot{G}_i = -uG_i + Lg_i\,, \tag{51}$$

where $L = \beta n(n-2)\,\gamma(t-\tau)\, h$. G_i depends only on h_i, since h is independent of the distribution of the h_j's, by (48).

We now transform (57) into an equation for g_i alone. First we derive an equation for $\lambda_i = h_i/h$, which is by (48) and (57)

$$\begin{aligned}
\dot{\lambda}_i &= \frac{1}{h}\left(\dot{h}_i - h_i \frac{\dot{h}}{h}\right)\\
&= \frac{1}{h}\left[-\alpha h_i + \frac{\beta\gamma(t-\tau)\, G_i}{(n-1)^2\delta} + \frac{\beta}{n-1}\, h^{(i)}(t-\tau) - h_i\left(-\alpha + \frac{\beta h(t-\tau)}{h}\right)\right]\\
&= -\beta\,\frac{h(t-\tau)}{h}\,\lambda_i + \frac{\beta\gamma(t-\tau)}{(n-1)^2 h\delta}\, G_i + \frac{\beta h(t-\tau)}{(n-1)\, h}\,\lambda^{(i)}(t-\tau),
\end{aligned}$$

where $\lambda^{(i)} = h^{(i)}/h$. Since $\lambda^{(i)} = 1 - \lambda_i$,

$$\dot{\lambda}_i = -\frac{\beta h(t-\tau)}{h}\left(\lambda_i - \frac{1-\lambda_i(t-\tau)}{n-1}\right) + \frac{\beta\gamma(t-\tau)}{(n-1)^2\delta h}\, G_i\,.$$

From this equation, we readily complete our transformation using the facts

$$\lambda_i - \frac{1-\lambda_i(t-\tau)}{n-1} = g_i + \frac{1}{n-1}\, g_i(t-\tau)$$

and $\dot{\lambda}_i = \dot{g}_i$. Thus,

$$\dot{g}_i = Ig_i + Jg_i(t-\tau) + KG_i\,, \tag{50}$$

where $I = \beta\gamma(t-\tau)/(n-1)^2\, h\delta$. Since, by (51), G_i depends only on g_i, we have indeed uncoupled the h_i's from the h_{jk}'s.

(IV) *A Second Order Equation for* g_i.

In order to eliminate G_i from (50), we differentiate (50) and then use (50) and (51) to eliminate terms involving G_i and $\dot{G}_i$.

This computation is straightforward and yields

$$\ddot{g}_i + A(t)\,\dot{g}_i + B(t)\,\dot{g}_i(t-\tau) + C(t)\, g_i + D(t)\, g_i(t-\tau) = 0, \tag{52}$$

where

$$\begin{aligned}
A(t) &= -I(t) + u - \dot{K}(t)\, K^{-1}(t),\\
B(t) &= -J(t),\\
C(t) &= -\dot{I}(t) - K(t)\, L(t) - I(t)(u - \dot{K}(t)\, K^{-1}(t)),
\end{aligned}$$

and

$$D(t) = -\dot{J}(t) - J(t)(u - \dot{K}(t)\, K^{-1}(t)).$$

In terms of the functions h, γ, and δ, these expressions become

$$A(t) = \frac{\beta h(t-\tau)}{h} + u + \frac{d}{dt} \log \frac{h\delta}{\gamma(t-\tau)},$$

$$B(t) = \frac{\beta}{n-1} \frac{h(t-\tau)}{h},$$

$$C(t) = \beta \left[\left(\frac{h(t-\tau)}{h} \right)^{\cdot} - \frac{\beta n(n-2)\, \gamma^2(t-\tau)}{(n-1)^2 \delta} + \frac{h(t-\tau)}{h} \left(u + \frac{d}{dt} \log \frac{h\delta}{\gamma(t-\tau)} \right) \right],$$

and

$$D(t) = \frac{\beta}{n-1} \left[\left(\frac{h(t-\tau)}{h} \right)^{\cdot} + \left(\frac{h(t-\tau)}{h} \right) \left(u + \frac{d}{dt} \log \frac{h\delta}{\gamma(t-\tau)} \right) \right].$$

Since $K_\tau(h) \neq 0$ and $K_\tau(\gamma) \neq 0$, we can invoke Lemma 1 to conclude that the limits A, B, C, and D of $A(t)$, $B(t)$, $C(t)$, and $D(t)$, respectively, exist as $t \to \infty$. To evaluate these limits, we let $k(\tau) = \beta e^{-\tau s(\tau)}$ and $\theta = 1/(n-1)$ for notational simplicity.

Then

$$\begin{aligned} A &= k(\tau) + u + (-u + \sigma(\tau)) \\ &= k(\tau) + \sigma(\tau), \\ B &= \theta k(\tau), \\ C &= \beta \left[0 - \frac{n(n-2)}{(n-1)^2} \sigma(\tau)\, e^{-\tau s(\tau)} + e^{-\tau s(\tau)}(u - u + \sigma(\tau)) \right] \\ &= k(\tau)\, \sigma(\tau) \left(1 - \frac{n(n-2)}{(n-1)^2} \right) \\ &= \theta^2 k(\tau)\, \sigma(\tau), \end{aligned}$$

and

$$\begin{aligned} D &= \frac{\beta \sigma(\tau)}{n-1} e^{-\tau s(\tau)} \\ &= \theta k(\tau)\, \sigma(\tau). \end{aligned}$$

Because the limits A, B, C, and D exist, it is natural to try to compare the behavior of g_i for large t with the behavior of w_i, where

$$\ddot{w}_i + A\dot{w}_i + B\dot{w}_i(t-\tau) + Cw_i + Dw_i(t-\tau) = 0 \tag{53}$$

For technical reasons, we instead compare the functions $\xi_i = g_i e^{\lambda t}$ and $\eta_i = w_i e^{\lambda t}$, where λ is a sufficiently small positive constant. To derive an equation for ξ_i, multiply (52) by $e^{\lambda t}$ and use the equalities

$$\ddot{g}_i e^{\lambda t} = \ddot{\xi}_i - 2\lambda \dot{\xi}_i + \lambda^2 \xi_i$$

and

$$\dot{g}_i e^{\lambda t} = \dot{\xi}_i - \lambda \xi_i .$$

We find

$$\ddot{\xi}_i + \bar{A}(t)\, \dot{\xi}_i + \bar{B}(t)\, \dot{\xi}_i(t-\tau) + \bar{C}(t)\, \xi_i + \bar{D}(t)\, \xi_i(t-\tau) = 0, \tag{54}$$

where

$$\begin{aligned} \bar{A}(t) &= A(t) - 2\lambda, \\ \bar{B}(t) &= B(t), \\ \bar{C}(t) &= C(t) + \lambda^2 - \lambda A(t), \end{aligned}$$

and

$$\bar{D}(t) = D(t) - \lambda B(t).$$

Similarly, η_i obeys the equation

$$\ddot{\eta}_i + \bar{A}\dot{\eta}_i + \bar{B}\dot{\eta}_i(t-\tau) + \bar{C}\eta_i + \bar{D}\eta_i(t-\tau) = 0, \tag{55}$$

where

$$\begin{aligned} \bar{A} &= A - 2\lambda \\ &= k + \sigma - 2\lambda \\ \bar{B} &= B \\ &= \theta k, \\ \bar{C} &= C + \lambda^2 - \lambda A \\ &= \theta^2 k\sigma + \lambda^2 - \lambda(k + \sigma), \end{aligned}$$

and

$$\begin{aligned} \bar{D} &= D - \lambda B \\ &= \theta k(\sigma - \lambda). \end{aligned}$$

If we can show that ξ_i is a bounded function, say $|\xi_i| \leqslant k$, then $|g_i| \leqslant ke^{-\lambda t}$ and since $\lambda > 0$,

$$g_i \equiv \frac{h_i}{\sum_{k=1}^{n} h_k} - \frac{1}{n} = O(e^{-\lambda t}), \tag{58}$$

which is the first claim of the theorem. To do this, we wish to compare the behavior of ξ_i for large t with that of η_i. If we can do this rigorously and if η_i is a bounded function, our proof of (58) will be complete. We now show that this comparison can be carried out and that the boundedness of η_i can be guaranteed by showing that there exists at least one positive λ such that all zeros of the characteristic exponential polynomial

$$G_{\beta n\tau}^{(\lambda)}(s) = s^2 + \bar{A}s + (\bar{B}s + \bar{D})\, e^{-\tau s} + \bar{C} \tag{56}$$

of (55) have negative real parts.

The theorems which we will need to accomplish these aims are applicable to (54) when it is written in the matrix form

$$\dot{Z}_i + (V_0 + V_0(t))\, Z_i + (V_1 + V_1(t))\, Z_i(t-\tau) = 0, \tag{59}$$

where

$$Z_i = \begin{pmatrix} \xi_i \\ \cdot \\ \xi_i \end{pmatrix}, \qquad V_0 = \begin{pmatrix} 0 & -1 \\ \bar{C} & \bar{A} \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 0 & 0 \\ \bar{D} & \bar{B} \end{pmatrix},$$

$$V_0(t) = \begin{pmatrix} 0 & 0 \\ \bar{C}(t) - \bar{C} & \bar{A}(t) - \bar{A} \end{pmatrix}, \quad \text{and} \quad V_1(t) = \begin{pmatrix} 0 & 0 \\ \bar{D}(t) - \bar{D} & \bar{B}(t) - \bar{B} \end{pmatrix}.$$

(55) has the matrix form

$$\dot{W}_i + V_0 W_i + V_1 W_i(t-\tau) = 0, \tag{60}$$

where $W_i = \binom{\eta_i}{\dot{\eta}_i}$. The first theorem which we shall need is the following ([5], p. 312).

A sufficient condition in order that all continuous solutions of (59) be bounded as $t \to \infty$ is that all solutions of (60) be bounded as $t \to \infty$, and that

$$\int_{t_0}^{\infty} \| V_i(t) \| \, dt < \infty, \qquad i = 0, 1,$$

for some $t_0 > 0$. The integrals $\int_{t_0}^{\infty} \| V_i(t)\| \, dt$ are certainly finite for sufficiently large t_0 by the inequalities (26)–(29) of Lemma 1. It therefore remains only to show that all solutions of (60) are bounded. We will be able to show more than this. In fact, by [5], p. 190, all solutions of (60) with sufficiently smooth initial data converge to zero as $t \to \infty$ iff all zeros of $G_{\beta n\tau}^{(\lambda)}(s)$ have negative real parts. We now show that all zeros of $G_{\beta n\tau}^{(\lambda)}(s)$ have negative real parts for a suitable choice of β, n, τ, and λ.

(V) *The Zeros of* $G_{\beta n\tau}^{(\lambda)}(s)$.

We will show that all zeros of $G_{\beta n\tau}^{(\lambda)}(s)$ have negative real parts if

$$k(\tau) + \sigma(\tau) > \frac{1}{n-1} k(\tau)(1 + \tau\sigma(\tau)) \qquad \text{and} \qquad \lambda > 0$$

is chosen sufficiently small. This fact relies on the following lemma.

LEMMA 2. *Suppose that the coefficients of the exponential polynomial*

$$G_{\beta n\tau}^{(\lambda)}(s) = s^2 + \bar{A}s + (\bar{B}s + \bar{D})\, e^{-\tau s} + \bar{C}$$

are positive and $\bar{A} > \bar{B} + \tau\bar{D}$. *Then all zeros of* $G_{\beta n\tau}^{(\lambda)}(s)$ *have negative real parts.*

Proof. The proof closely follows [7], in which the closely related exponential polynomial

$$az^2 + bz + \beta z e^{-z} + c$$

is studied. Letting $z = \tau s$, $\tau > 0$, the equation $G_{\beta n\tau}^{(\lambda)}(s) = 0$ becomes

$$f(z) \equiv z^2 + Ez + (Fz + H)\, e^{-z} + J = 0,$$

where $E = \bar{A}\tau$, $F = \bar{B}\tau$, $H = \bar{D}\tau^2$, and $J = \bar{C}\tau^2$. The zeros of $f(z)$ are the same as the zeros of $G_{\beta n\tau}^{(\lambda)}(s)$ for all $\tau > 0$. For $\tau = 0$, it is obvious that all zeros of $G_{\beta n\tau}^{(\lambda)}(s) = 0$ have negative real parts if λ is chosen sufficiently small, by the positivity of $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$. In the following, we therefore consider the zeros of $f(z)$ for $\tau > 0$. In this case, E, F, H, and J are all positive if λ is sufficiently small.

The main fact used in our analysis is Cauchy's Index Theorem : Suppose $w = f(z)$ is an analytic function of z in a simply connected domain D bounded by a closed curve C, where $f(z) \neq 0$ for $z \in C$. If z traverses C in a counterclockwise direction, then $f(z)$ will traverse a closed curve in the w-plane and the number of zeros of $f(z)$ in D is equal to the number of times the w-contour encircles the origin.

The zeros of $w = f(z)$ are studied using this theorem in the following way. As z traverses C in a counterclockwise direction, w may cross the real axis. Let δ_+ be the number of times that w crosses the real axis in a counterclockwise direction relative to the origin (i.e., from quadrant IV to quadrant I or from quadrant II to quadrant III), and let δ_- be the number of times w

crosses the real axis in a clockwise direction relative to the origin. The number of zeros of $f(z)$ in D then equals $\frac{1}{2}(\delta_+ - \delta_-)$.

We apply Cauchy's Index Theorem to the semicircular domain

$$D : \text{Re}(z) > 0 \quad \text{and} \quad |z| < R$$

in the z-plane. If $|z| \geqslant R$ and $\text{Re}(z) \geqslant 0$, where R is chosen sufficiently large, then

$$|z^2| > |Ez + (Fz + H)e^{-z} + J|$$

and $z^2 \neq 0$. Rouche's theorem therefore implies that $f(z) \neq 0$ for $|z| \geqslant R$ and $\text{Re}(z) \geqslant 0$. D is fixed once and for all by making a choice of a sufficiently large R. For this choice of D, all the zeros of $f(z)$ in the right half plane will lie in D.

To apply Cauchy's theorem to this domain D, we divide its boundary curve C into two parts

$$C_1 : \text{Re}(z) = 0 \quad \text{and} \quad |z| \leqslant R$$

and

$$C_2 : \text{Re}(z) > 0 \quad \text{and} \quad |z| = R.$$

Consider C_1. Then $z = iy$ and

$$f(iy) = -y^2 + H\cos y + Fy\sin y + J$$
$$+ i(Ey + Fy\cos y - H\sin y),$$

where $J > 0$ and $E \geqslant F + H$ by hypothesis. If $y = 0$, then $f(0) = H + J > 0$. If $0 < y \leqslant R$, then

$$\text{Im}(f(iy)) = y\left(E + F\cos y - H\frac{\sin y}{y}\right)$$
$$\geqslant 0,$$

since $|\sin y/y| \leqslant 1$. Thus w is in either quadrant I or quadrant II. $f(iR)$ is in quadrant II if R is sufficiently large. If $-R \leqslant y < 0$, then $\text{Im}(f(iy)) \leqslant 0$ and w is in either quadrant III or quadrant IV. Since also $f(0)$ is a positive real number, as z traverses C_1 from $+iR$ to $-iR$, w crosses the real axis once in a clockwise direction relative to the origin.

Now consider $f(z)$ as z traverses C_2 from $-iR$ to $+iR$. Since R is large, $f(z)$ behaves essentially like z^2. Thus the net number of times that $f(z)$ crosses the real axis in a counterclockwise direction relative to the origin is once. We have therefore shown that $\frac{1}{2}(\delta_+ - \delta_-) = 0$, and that $f(z)$ has no zeros in D, or for that matter in the right half plane. This completes the proof of the Lemma.

We remark that [7] goes on to give necessary and sufficient conditions for his equation to have zeros only in the left half plane even when the inequalities analogous to $J > 0$ and $E \geqslant F + H$ are not satisfied. These conditions seem to be difficult to apply in the present case.

We now apply Lemma 2 to the present example. When $\tau = 0$, the zeros of $G^{(\lambda)}_{\beta n\tau}(s)$ obviously have negative real parts for sufficiently small $\lambda > 0$ by the positivity of $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$. Consider the case $\tau > 0$, where λ is chosen so small that $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ are all positive. The condition $J > 0$ is satisfied since $\bar{C} > 0$ and $\tau > 0$. The condition $E \geqslant F + H$ becomes $\bar{A}\tau \geqslant \bar{B}\tau + \bar{D}\tau^2$ or $\bar{A} \geqslant \bar{B} + \tau\bar{D}$. Since

$$\bar{A} = k(\tau) + \sigma(\tau) - 2\lambda, \ \bar{B} = \theta k(\tau), \quad \text{and} \quad \bar{D} = \theta k(\tau)(\sigma(\tau) - \lambda),$$

this inequality becomes

$$k(\tau) + \sigma(\tau) - 2\lambda \geqslant \frac{1}{n-1} k(\tau)[1 + \tau(\sigma(\tau) - \lambda)].$$

If, as hypothesized, $k(\tau) + \sigma(\tau) > (n-1)^{-1} k(\tau)(1 + \tau\sigma(\tau))$, we can certainly find a sufficiently small positive λ for which $E \geqslant F + H$. This shows that all the zeros of $G^{(\lambda)}_{\beta n\tau}(s)$ have negative real parts if

$$k(\tau) + \sigma(\tau) > (n-1)^{-1} k(\tau)(1 + \tau\sigma(\tau))$$

and λ is chosen sufficiently small. For $\lambda = \omega_1$ chosen in this way, we can therefore conclude that

$$\frac{h_i(t)}{\sum_{k=1}^{n} h_k(t)} - \frac{1}{n} = O(e^{-\omega_1 t})$$

This completes the first part of the proof.

Remark. The condition $\bar{A} \geqslant \bar{B} + \tau\bar{D}$ needed to guarantee the negativity of the real parts of $G^{(\lambda)}_{\beta n\tau}$'s zeros gets harder to fulfill as τ increases. When $\bar{A} < \bar{B} + \tau\bar{D}$, the condition needed to guarantee the same result contains oscillatory terms. No such difficulty arises in treating the variational system of (*), since all terms in the equation for g_i which contain the time lag τ are independent of the distribution of h_i's and h_{jk}'s.

(VI) *An equation for* $g_{jk} = h_{jk}[\sum_{m\neq j} h_{jm}] - (n-1)^{-1}$.

We now prove the existence of an $\omega_2 > 0$ for which

$$h_{jk}(t)\Big[\sum_{\substack{m=1\\m\neq j}}^{n} h_{jm}(t)\Big]^{-1} - \frac{1}{n-1} = O(e^{-\omega_2 t}), \tag{43}$$

for all $j \neq k$. This we do by deriving an equation for g_{jk} of the form

$$\dot{g}_{jk} = \Lambda_j(G_{jk} - g_{jk}), \tag{61}$$

where G_{jk} converges exponentially to zero as $t \to \infty$, Λ_j has the form

$$\Lambda_j = \frac{d}{dt} \log \Gamma_j , \tag{62}$$

and Γ_j satisfies suitable growth estimates as $t \to \infty$. We can treat (61) in the same way as we did (19), and therefore we merely exhibit the relevant formulas.

To derive (43), we first derive an equation for $H_{jk} = h_{jk}/H^{(j)}$, where $H^{(j)} = \sum_{m \neq j} h_{jm}$. By (47),

$$\dot{H}^{(j)} = -uH^{(j)} + \beta(n-1)\,\gamma h_j(t-\tau) + \beta\gamma(t-\tau)\,h^{(j)}. \tag{63}$$

Thus by (47) and (63),

$$\begin{aligned}\dot{H}_{jk} &= \frac{1}{H^{(j)}}\left[\dot{h}_{jk} - h_{jk}\frac{\dot{H}^{(j)}}{H^{(j)}}\right] \\ &= \frac{\beta}{H^{(j)}}\,[\gamma h_j(t-\tau) + \gamma(t-\tau)\,h_k \\ &\quad - H_{jk}((n-1)\,\gamma h_j(t-\tau) + \gamma(t-\tau)\,h^{(j)})],\end{aligned}$$

from which follows

$$\begin{aligned}\dot{g}_{jk} &= \dot{H}_{jk} \\ &= \frac{\beta}{H^{(j)}}\Big\{\gamma h_j(t-\tau) + \gamma(t-\tau)\,h_k - \left(H_{jk} - \frac{1}{n-1}\right) \\ &\quad \cdot [(n-1)\,\gamma h_j(t-\tau) + \gamma(t-\tau)\,h^{(j)}] \\ &\quad - \left[\gamma h_j(t-\tau) + \frac{1}{n-1}\gamma(t-\tau)\,h^{(j)}\right]\Big\} \\ &= \frac{\beta}{H^{(j)}}\Big\{\gamma(t-\tau)\left[h_k - \frac{1}{n-1}h^{(j)}\right] \\ &\quad - g_{jk}[(n-1)\,\gamma h_j(t-\tau) + \gamma(t-\tau)\,h^{(j)}]\Big\}.\end{aligned}$$

Letting

$$\Lambda_j = \frac{\beta[(n-1)\,\gamma h_j(t-\tau) + \gamma(t-\tau)\,h^{(j)}]}{H^{(j)}}$$

and

$$G_{jk} = \frac{\gamma(t-\tau)\left[h_k - \frac{1}{n-1}h^{(j)}\right]}{(n-1)\,\gamma h_j(t-\tau) + \gamma(t-\tau)\,h^{(j)}}, \tag{64}$$

we find (61). (62) is also true, because $\Lambda_j = d/dt \log \Gamma_j$, where

$$\Gamma_j(t) = H^{(j)}(0) + \beta\int_0^t e^{uv}[(n-1)\,\gamma h_j(v-\tau) + \gamma(v-\tau)\,h^{(j)}]\,dv. \tag{65}$$

To show that G_{jk} converges exponentially to zero as $t \to \infty$, divide numerator and denominator of (64) by $\gamma(t-\tau)\,h$. Then

$$\begin{aligned}G_{jk} &= \frac{\lambda_k - \frac{1}{n-1}(1-\lambda_j)}{(n-1)\frac{\gamma h(t-\tau)}{\gamma(t-\tau)h}\lambda_j(t-\tau) + (1-\lambda_j)} \\ &= \frac{g_k + \frac{1}{n-1}g_j}{(n-1)\frac{\gamma h(t-\tau)}{\gamma(t-\tau)h}\left(g_j(t-\tau) + \frac{1}{n}\right) + \frac{n-1}{n} - g_j}\end{aligned}$$

Since g_j and g_k converge exponentially to zero as $t \to \infty$, whereas $(\gamma h(t-\tau)/\gamma(t-\tau)\,h)$ converges to 1, the result is proved.

It is readily seen, just as in step (V) of Theorem 3, that Γ_j can be written in the form

$$\Gamma_j(t) = H^{(j)}(0) + e^{\sigma(\tau)t}\left[\frac{\mu}{\sigma(\tau)}(1 - e^{-\sigma(\tau)t}) + e^{-kt}R_j(t)\right],$$

where $\mu \neq 0$, $k > 0$, and R_j is bounded. (43) then follows in the same way as in the proof of Theorem 3. This establishes Theorem 4.

References

1. Grossberg, S., A prediction theory for some nonlinear functional-differential equations (I). *J. Math. Anal. Applics.* **21** (1968), 643–693.
2. Grossberg, S., On the global limits and oscillations of a system of nonlinear differential equations describing a flow on a probabilistic network, *J. Diff. Eqns.* **5** (1969), 531–563.
3. Grossberg, S., A prediction theory for some nonlinear functional-differential equations (II), *J. Math. Anal. Applics.* **22** (1969), 490–522.
4. Halanay, A., "Differential Equations; Stability, Oscillations, Time Lags." Academic Press, New York, 1966.
5. Bellman, R. E. and Cooke, K. L., "Differential-Difference Equations." Academic Press, New York, 1963.
6. Bellman, R. R., "Stability Theory of Differential Equations." McGraw-Hill, New York, 1953.
7. Sherman, S., A note on stability calculations and time lag. *Quart. Applied Math.* **5** (1947), 92.

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