Julia Sets Converging to the Filled Basilica

Robert Thÿs Kozma
Professor: Robert L. Devaney

Boston University

Young Mathematicians Conference
Columbus OH, August 29, 2010

Outline

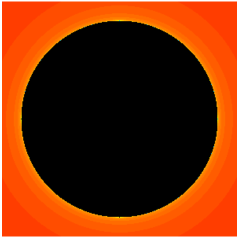
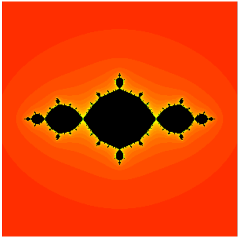
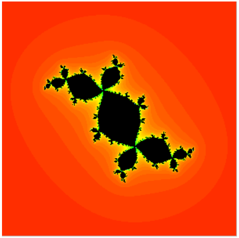
1. **Introduction**
   Basic concepts of dynamical systems
2. **Theoretical Background**
   Perturbations of quadratic maps
3. **Main Results**
   Proof of the convergence of a family of Julia sets to the filled basilica
4. **Concluding Remarks**
   Conjectures for future studies

Preliminaries of the Problem

Complex quadratic function $\mathbf{F(z) = z^2 + c}$, (unperturbed)

Iterates $F^2(z) = F(F(z)), \ldots, F^n(z) = F(F^{n-1}(z)), \ldots$

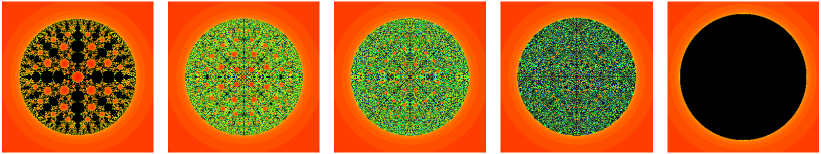
Black: $F^n(z)$ remains bounded. Colored: $F^n(z) \to \infty$ as $n \to \infty$.

Unit disk	Basilica	Douady Rabbit
		
$c = 0$	$c = -1$	$c \approx -0.123 + 0.745i$

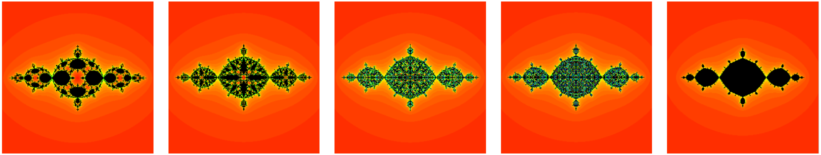
The Julia set is the boundary of the black region.

Perturbed Quadratic Map (PQM) $F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$

<u>Previous result '08:</u> (Julia set of PQM with $c = 0$) $\xrightarrow[\lambda \to 0]{}$ unit disk



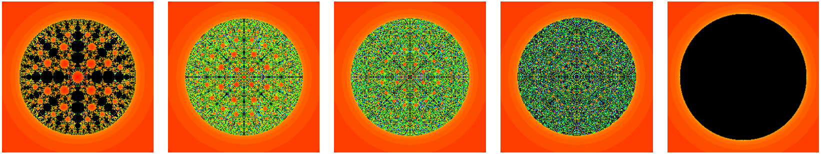
<u>Present work:</u> (Julia set of PQM with $c = -1$) $\xrightarrow[\lambda \to 0+]{}$ filled basilica



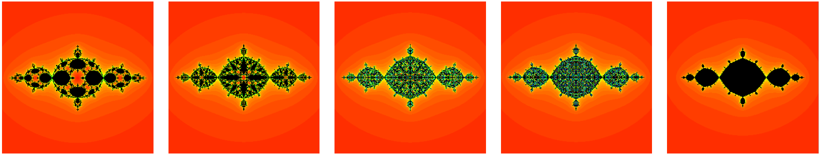
$\lambda = 10^{-2}$ $\lambda = 10^{-3}$ $\lambda = 10^{-4}$ $\lambda = 10^{-6}$ $\lambda = 0$

Perturbed Quadratic Map (PQM) $F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$

<u>Previous result '08:</u> (Julia set of PQM with $c = 0$) $\xrightarrow[\lambda \to 0]{}$ unit disk



<u>Present work:</u> (Julia set of PQM with $c = -1$) $\xrightarrow[\lambda \to 0+]{}$ filled basilica

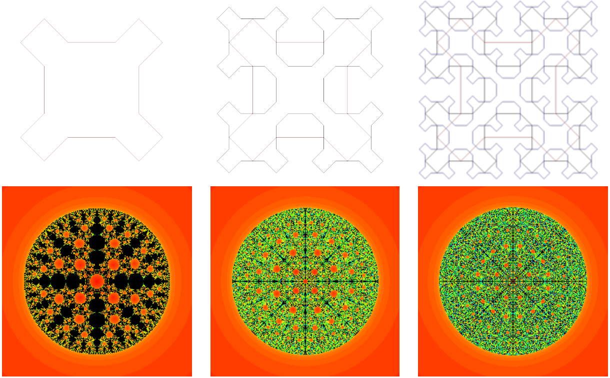


$\lambda = 10^{-2}$ $\lambda = 10^{-3}$ $\lambda = 10^{-4}$ $\lambda = 10^{-6}$ $\lambda = 0$

<u>Conjecture:</u> Julia set of PQM with c in the center of any periodic bulb in the Mandelbrot set $\xrightarrow[\lambda \to 0]{}$ filled Julia set.

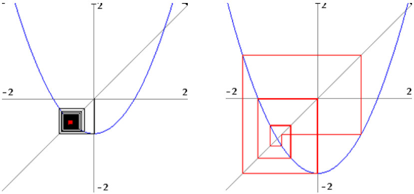
On Convergence

Space-filling curve (Sierpinski curve)



Iterates and Orbits of Quadratic maps $z^2 + c$

Graphical Analysis



$-\frac{3}{4} < c < \frac{1}{4}$ $\quad$ $c \approx -1.5$

Orbit of 0 when $F(z) = z^2 + c$

1. $F(0) = 0^2 + c$
2. $F(F(0)) = F(c) = c^2 + c$
3. Do it again: $F(F(F(0))) = F(c^2 + c) = (c^2 + c)^2 + c$
4. Do it again ...

$\underbrace{F(F(\ldots F}_{n}(0)..)$ is called the ***n*-th iterate** and denoted $F^n(0)$. The list of numbers generated in this way called the **orbit of 0**.

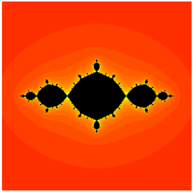
The Julia Set

Given $c \in \mathbb{C}$ the **filled Julia set** for $z^2 + c$ is the collection of all z's whose orbit does not escape to infinity under iteration of $z^2 + c$.

Definition

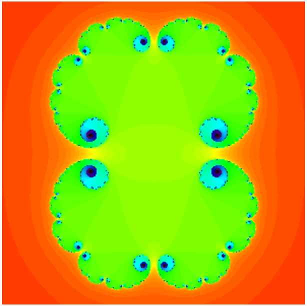
The ***Julia set*** *is the boundary of the filled Julia set and is denoted* $J(F)$.

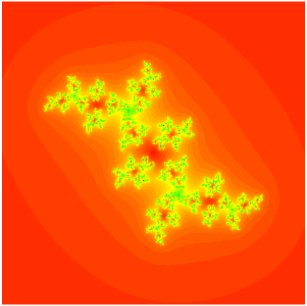
The Basilica



$c = -1$

Other Julia sets can look very different



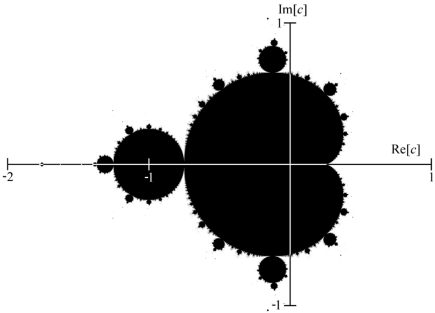


Julia sets are either connected or totally disconnected (black region is a scattered set of points).

The Mandelbrot Set

Definition

The ***Mandelbrot set*** $\mathcal{M}$ *consists of all of all* $c \in \mathbb{C}$ *values for which the filled Julia set* $J(z^2 + c)$ *is connected.*



Perturbed Quadratic Maps

Perturbed complex quadratic polynomial maps are functions of the form

$$F_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^2} \text{ with } \lambda \in \mathbb{C}.$$

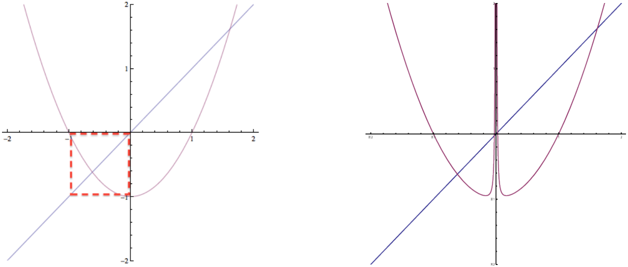


Illustration of unperturbed and perturbed quadratic maps, $\lambda \in \mathbb{R}^+$.

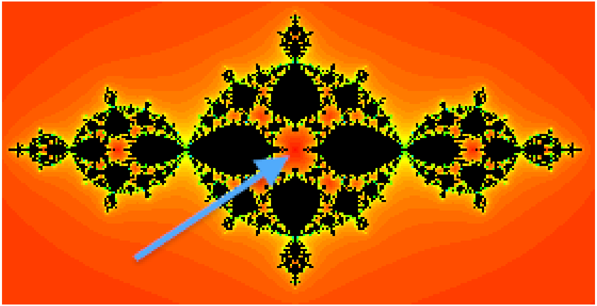
Trap Door

B_λ is the immediate basin of attraction of ∞ (outer red region).

Definition

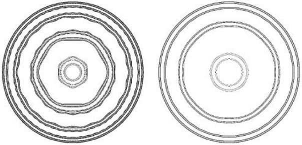
The region about the origin which is mapped to B_λ under one iteration is called the ***trap door*** *and is denoted T_λ.*

All colored regions within ∂B_λ are preimages of the T_λ.

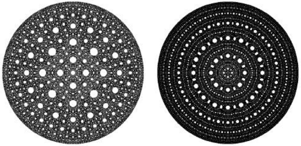


Julia sets converge to unit disk (Garijo & Devaney, 2008)

$H_\lambda(z) = z^n + \lambda/z^d$ in the case of $n, d > 2$ there always exists an annulus of fixed size in the complement of the Julia set for all λ ...



... but $H_\lambda(z) = z^n + \lambda/z^d$ converges to the unit disk in the case of $n = d = 2$ as $\lambda \to 0$.



Main Theorem:
The Julia set of $F_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^2}$ converges to the filled basilica for parameter values $\lambda \in \mathbb{R}^+$ as $\lambda \to 0$ when $\lambda \neq 0$.

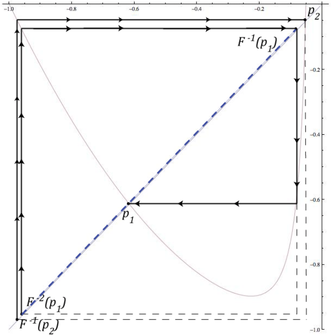
Steps in Proof:

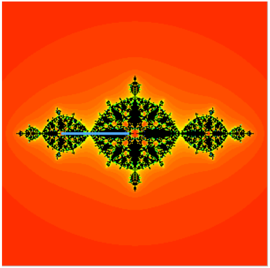
1. Find an invariant interval in the Julia set
2. Behavior of the 2nd iterate of the central bulb.
3. Construction of a Cantor necklace
4. Finish by showing that iterates of a disk of radius ϵ in the central bulb will intersect with the Cantor necklace in $J(F_\lambda(z))$.

The Invariant Interval/1

Proposition

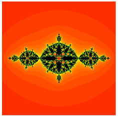
There is an invariant interval I in the filled Julia set under the map F_λ^2 which connects T_λ and the preimage of T_λ in D_1.

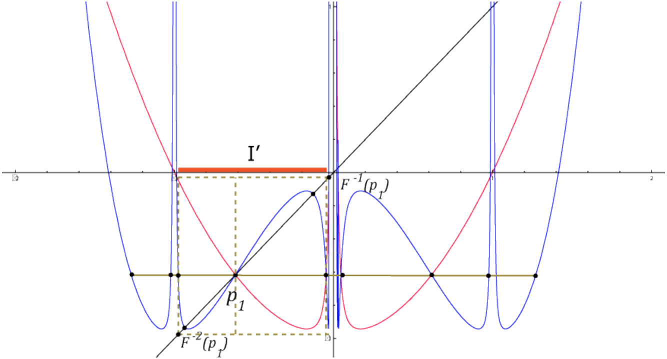




Graphical analysis of the first and second iterate of $F_\lambda(z)$.

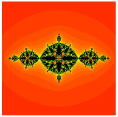
The Invariant Interval/2

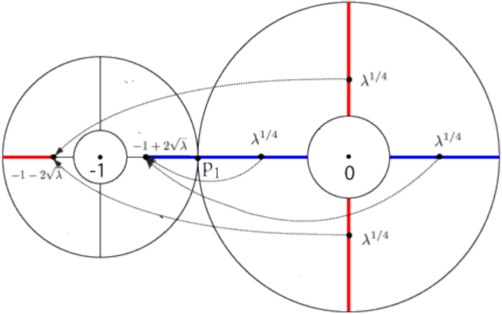




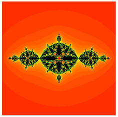
Graphical analysis of the first and second iterate of $F_\lambda(z)$. We show there is a invariant interval under the second iterate.

The Invariant Interval/3

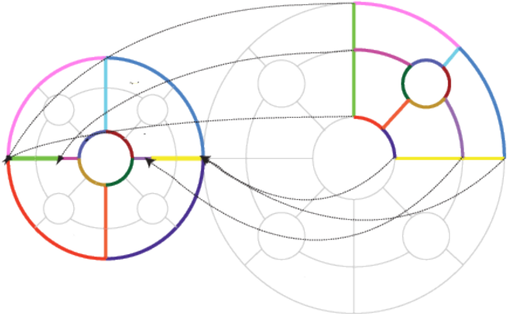




Schematic illustration of two regions in the basilica Julia set. Segments a given same color are mapped to the segments of the same color.

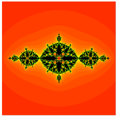


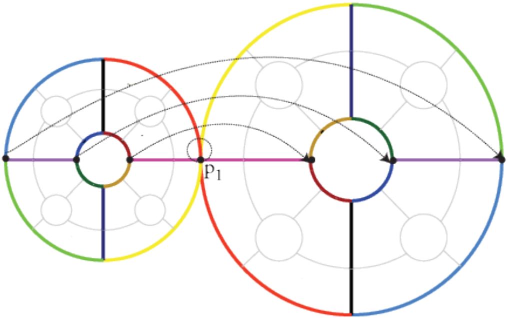
Behavior of the 1st Iterate of a Sector



Segments a given same color are mapped to the segments of the same color.

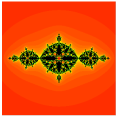
Behavior of the 2nd Iterate of a Sector/1

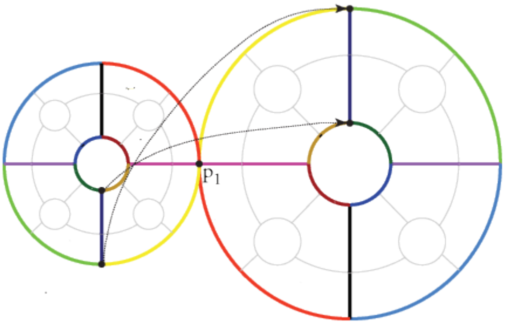




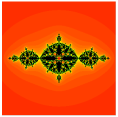
Segments a given same color are mapped to the segments of the same color.

Behavior of the 2nd Iterate of a Sector/2

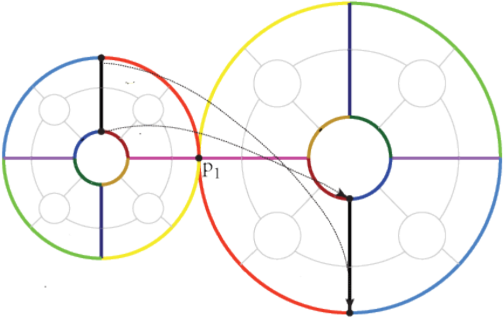




Segments a given same color are mapped to the segments of the same color.



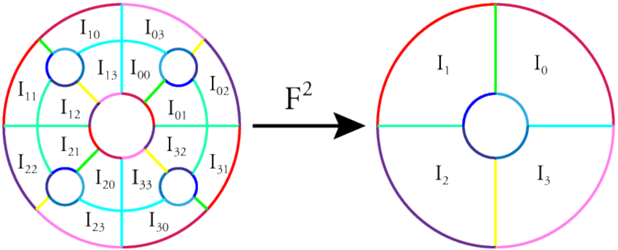
Behavior of the 2nd Iterate of a Sector/3



Segments a given same color are mapped to the segments of the same color.

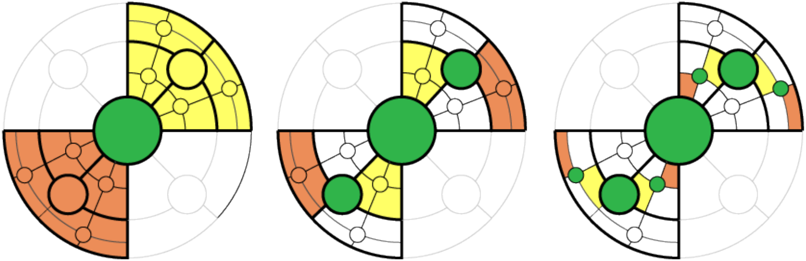
F_λ^2 maps the central bulb of the basilica onto itself

The mapping of central bulb D_0 under F_λ^2 is 4-to-1.



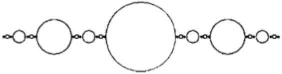
The 4 preimages of each of the fundamental sectors of the central bulb. Its four preimages in $J(F_\lambda)$ under F_λ^2 are shown. Segments of the same color are mapped to the segments of the same color.

1st, 2nd, ... preimages of 2 central bulb sectors of J



Green: preimages of B_λ; Yellow and Brown sectors: trace the evolution of the preimages.

This is homeomorphic to the Cantor middle thirds necklace;



generalization of Devaney (2006).

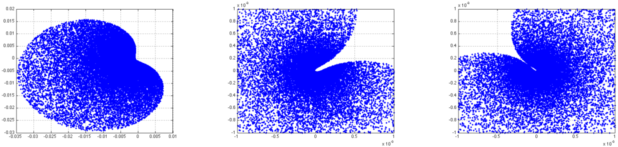
Final Step of the Proof

Theorem (Disk Wrapping around the origin)

Let $B_\epsilon(z_0)$ denote the disk of radius $\epsilon > 0$ centered at z_0. There exists a $\mu > 0$ such that, for any $\lambda \in \mathbb{R}^+$ such that $0 < |\lambda| \leq \mu$, $J(F_\lambda) \cap B_\epsilon(z_0) \neq \emptyset$ for all $z_0 \in D_0$.

- The $2k$-th iterates of disk always stays in central bulb D_0.
- $\exists n, \ F^{2n}(B_\epsilon)$ wraps around origin, intersects Cantor necklace.
- Julia set is backwards invariant.
- This concludes the proof of the main result. $\square$

Julia set of F_λ for $\lambda \to 0+$ converges to the filled basilica.



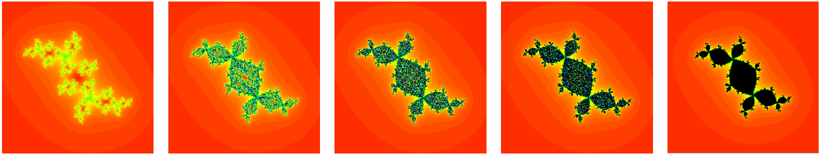
Conclusion & Future Directions

Conjecture

The Julia set of

$$F_\lambda = z^2 + c + \frac{\lambda}{z^2}$$

converges to the filled Julia set given that the value c lies in the center of a hyperbolic component of the Mandelbrot set.



$\lambda = 10^{-2}$ $\lambda = 10^{-3}$ $\lambda = 10^{-4}$ $\lambda = 10^{-6}$ $\lambda = 0$

References

- Devaney, R.L., *Cantor Necklaces and Structurally Unstable Sierpinski Curve Julia Sets for Rational Maps*, Qualitative Theory of Dynamical Systems 5 (2006), 337-359.
- Devaney, R.L., Garijo, A. *Julia Sets Converging to the Unit Disk*, Proc. AMS, Vol. 136 (2008) 981-988.
- Morabito, M., Devaney, R.L., *Limiting Julia Sets for Singularly Perturbed Rational Maps*, International Journal of Bifurcation and Chaos 18 (2008), 3175-3181.
- Blanchard, P., Devaney, R.L., Garijo, A., Marotta, S.M., Russel E. D. *Rabbits, Basilicas, and Other Julia Sets Wrapped in Sierpinski Carpets*, Complex Dynamics: Families and Friends, A. K. Peters (2009), 277-296.
- Kozma, R.T., *Julia Sets Converging to the Filled Basilica*, *PiMuEpsilon Journal* (Accepted, 2010).