Julia Sets of Perturbed Quadratic Maps Converging to the Filled Basilica

Robert Thijs Kozma*

July 21, 2010

Abstract

In this paper we investigate singularly perturbed complex quadratic polynomial maps of the form

$$F_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^2}.$$

We prove that for parameter values $\lambda \in \mathbb{R}^+$ as $\lambda \to 0$ the Julia set of $F_\lambda(z)$ converges to the filled basilica when $\lambda \neq 0$.

1 Introduction

Our goal in this paper is to show that the Julia sets of the functions $z^2 - 1 + \lambda/z^2$ converge to the filled basilica, the filled Julia set of $z^2 - 1$, as $\lambda \to 0$ along the positive real axis. Julia sets of singularly perturbed polynomial maps from the family $H_\lambda(z) = z^n + c + \lambda/z^d$ have been extensively studied in recent years. These maps are obtained by replacing the only finite critical point of $H_0(z)$, the z_0 value for which the derivative is zero, by a pole of order n.

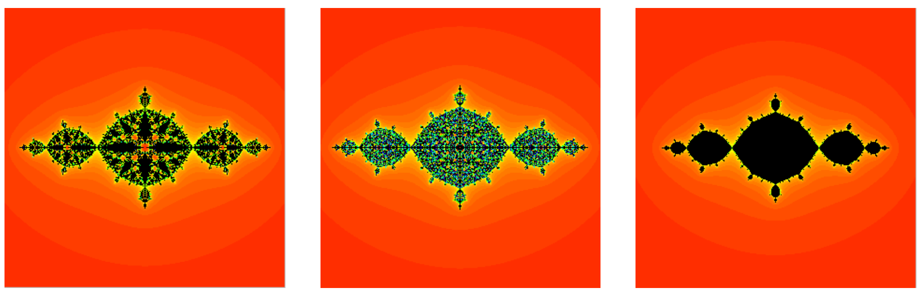
For case $c = 0$, is well known that the filled Julia set of $H_0(z) = z^n$ is the unit disk [8]. The family of maps $H_\lambda(z) = z^n + \lambda/z^d$ was investigated for $d = 1$ in [6], and for $n, d \geq 2$ in [3]. For family H_λ the Julia sets behave in three different ways as $\lambda \to 0$. In the case $d = 1$, the Julia set converges to the closed unit disk as $\lambda \to 0$ along $n - 1$ special rays in $\mathbb{C}$. For $n = d = 2$ the Julia set converges to the unit disk as $\lambda \to 0$ from all directions in $\mathbb{C}$. In cases with $n, d > 2$, regardless of how small $|\lambda|$, Julia sets no longer converge, as there is always an annulus of fixed size about the origin in the complement of the Julia set.

In [1] the authors investigate the family $H_\lambda(z) = z^n + c + \lambda/z^d$ for various c values located at centers of hyperbolic components of the Mandelbrot set. It is shown that the Julia set explodes for $\lambda \neq 0$, and for cases where $c \neq 0$ the boundaries of the components of the basin of infinity are not simple closed

*Boston Univeristy

curves, but rather doubly inverted copies of the Julia set of $z^2 + c$. In this paper, we use the boundary structure of the Julia set to show that the Julia set of $F_\lambda(z) = z^2 - 1 + \lambda/z^2$ converges to the filled basilica.

This paper is structured as follows. In Section 2, we outline preliminaries on Julia sets. In Section 3, we show that the critical orbits of F_λ never escape to infinity for small $|\lambda|$, by finding an invariant interval in the filled Julia set under the second iterate of F_λ containing the critical orbit. The second iterate of F_λ is denoted F_λ^2. In Section 4, we show that the dynamics of F_λ^2 on a portion of the Julia set is conjugate to the shift map on two symbols, and use symbolic dynamics to construct a Cantor necklace in the dynamical plane. In Section 5, we conclude the paper by proving that the existence of the invariant Cantor necklace in the dynamical plane implies that the Julia set converges to the filled basilica as $\lambda \to 0$.



(i) (ii) (iii)

Figure 1: Filled Julia sets for maps (i) $F_{0.001}$; (ii) $F_{0.00001}$; (iii) F_0, called the filled basilica. Black points have bounded orbits, while colored points escape to infinity; the Julia set is the boundary of the colored region.

2 Preliminaries

The *Julia set* is the chaotic domain of a complex analytic function. Equivalently, the Julia set is the boundary of the basin of attraction of infinity, which for our maps is a superattracting fixed point. The complement of the Julia set is called the *Fatou set*. All attracting periodic orbits lie within the Fatou set, while the boundary of the Fatou set, as well as all repelling periodic points are contained in the Julia set. The Fatou set typically has simple dynamics, with most orbits converging to attracting cycles.

The filled Julia set consists of all points whose orbits are bounded. The Julia set denoted $J(F_\lambda)$ is the boundary of the filled Julia set. The Julia set of the unperturbed quadratic map $J(F_0)$ is called the basilica and is shown in Figure 1 (iii). We denote the immediate basin of attraction of infinity by B_λ. The region about the origin in the dynamical plane that is mapped to B_λ under one iteration is called the *trap door* and is denoted T_λ.

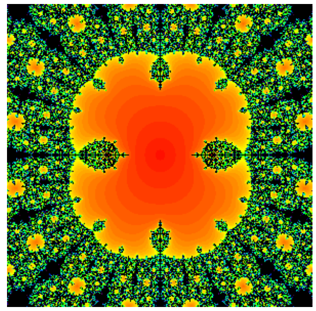


Figure 2: The filled Julia set of $F_{0.00001}$ enlarged about the trap door, with details of the doubly inverted basilicas.

It is known that T_λ and B_λ are disjoint when $|\lambda|$ is sufficiently small; in this paper we assume this to always be the case. The boundary of the immediate basin of attraction of infinity under F_λ is denoted ∂B_λ and is homeomorphic to $J(F_0)$. The map F_λ restricted to ∂B_λ is conjugate to F_0 on $J(F_0)$. However, the inner structure of $J(F_\lambda)$ is much more intricate: there are infinitely many doubly inverted copies of $J(F_0)$ for λ values near zero [1]. Examples of doubly inverted basilicas are given in Figure 2.

3 Critical Orbit Remains Bounded

In this section, we prove necessary conditions for $J(F_\lambda)$ converging to the filled basilica as $\lambda \to 0+$. The dynamical behavior of the critical points determine the structure of the Julia set. If the second iterates of the critical points lie in the trap door, then previous results would allow the possibility of an annuli in the Fatou set, which would not allow the Julia set to convegre to the filled basilica [3]. We show that the critical orbit remains bounded for all iterations of F_λ, by finding an invariant interval in the filled Julia set of F_λ extending between T_λ and its preimage under F_λ^2.

The four distinct critical points of F_λ are $c_\lambda = \lambda^{1/4}$. In the case of $\lambda \in \mathbb{R}$ the four c_λ are mapped to $-1 \pm 2\sqrt{\lambda}$. By simple analysis, it follows that $\lim_{\lambda\to 0} F_\lambda^3(c_\lambda) = -15/16$. This value lies a constant distance from the preimage of T_λ about -1 for $|\lambda|$ small enough.

We introduce the following notation to help in the discussion of Julia sets. Let D_0 denote the bulb of the Julia set of F_0 that contains 0. Similarly, let D_1 be the bulb of the Julia set about -1. The corresponding objects in the perturbed map F_λ are denoted the same.

We prove that the filled Julia set of F_λ contains a connected interval along the real axis running between T_λ, and its preimage in D_1. This is done by showing no preimages of T_λ are located in this interval. The orbits of all points within this interval, including $-15/16$, remain bounded for all iterations of F_λ.

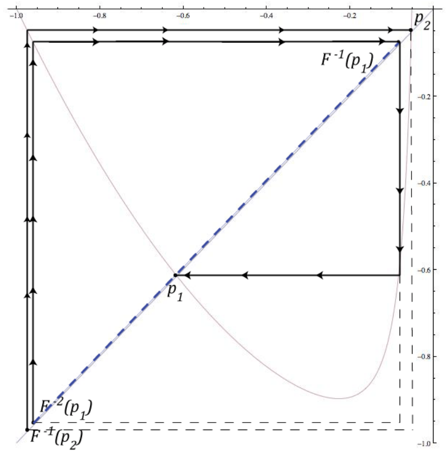


Figure 3: Graphical analysis of intervals $I = [F_\lambda^{-2}(p_1), F_\lambda^{-1}(p_1)]$ and $I' = [F_\lambda^{-1}(p_2(\lambda)), p_2(\lambda)]$. The dynamics of the featured section is similar to that of the regular quadratic map.

Proposition 1 *Let $p_2(\lambda)$ denote the negative fixed point of $F_\lambda(z)$ near zero. The interval $I' = [F_\lambda^{-1}(p_2(\lambda)), p_2(\lambda)]$ contains the local minima of the quadratic map, i.e., $F_\lambda^{-1}(p_2(\lambda)) < -1 + 2\sqrt{\lambda}$ for all λ.*

Proof: We obtain an algebraically closed form for $p_2(\lambda)$ and its preimage $F_\lambda^{-1}(p_2(\lambda))$ in terms of λ in 4 steps:

1. In order to find the fixed point, we solve the quartic equation $z^2 - 1 + \lambda/z^2 = z$ using Ferrari's method. Multiplying by z^2 and rearranging terms we obtain $z^4 - z^3 - z^2 + \lambda = 0$. Using Ferrari's method we transform our quartic into the cubic $y^3 + y^2 + 4\lambda y - 5\lambda = 0$.

2. We use Cardano's formula to solve the cubic and obtain a formula of the fixed point $p_2(\lambda)$ in terms of λ. The details are left to the reader.

3. We substitute the value of the fixed point $p_2(\lambda)$ into the biquadratic formula for the preimage of F_λ. Hence we obtain a closed form equation for $F_\lambda^{-1}(p_2(\lambda))$ with λ as the only parameter.

$$F^{-1}(\lambda) = -\sqrt{\frac{p_2(\lambda) + 1 + \sqrt{(-(p_2(\lambda)+1))^2 - 4\lambda}}{2}} \tag{1}$$

4. For small $|\lambda|$ values, we have that $F_\lambda^{-1}(p_2(\lambda)) < -1 + 2\sqrt{\lambda}$. Hence the local minimum of the perturbed quadratic map is contained in the interval $I' = [F_\lambda^{-1}(p_2(\lambda)), p_2(\lambda)]$ completing the proof.

QED

Note that solving with Ferrari's method and Cardano's formula require no numerical methods so the proof of Proposition 1 is mathematically rigorous.

Let $p_1 = (1 - \sqrt{5})/2$ denote the fixed point of the unperturbed quadratic map. Let $I = [F_\lambda^{-2}(p_1), F_\lambda^{-1}(p_1)]$. In this case $I \subseteq I'$, see Figure 3.

Corollary 1 *As $\lambda \to 0$, the closed inteval I contains one of the two local minima of F_λ along the real axis.*

By graphical analysis of F_λ over the interval I given in Figure 3, we find that the interval I is invariant under F_λ.

Corollary 2 *F_λ maps the closed interval I 2-to-1 onto itself.*

In the next proposition, we show the intervals I and I' both contain the segment connecting the T_λ to its preimage, by proving that $p_2(\lambda)$ lies on ∂T_λ, and that the I' interval is invariant under F_λ^2. As a result there is a connected piece in $J(F_\lambda)$ joining T_λ and its preimage in D_1. Moreover, we find that $p_2(\lambda) > F^{-1}(p_1)$ where $p_2(\lambda)$ is the negative fixed point near 0. One can find a closed form for the fixed point $p_1(\lambda)$, the fixed point near $p_1 = (1 - \sqrt{5})/2$, in terms of λ using the method outlined for $p_2(\lambda)$ in Proposition 1.

Proposition 2 *Two preimages of the fixed point $p_1(\lambda)$ lie at $\partial T_\lambda \cap \mathbb{R}$, and two lie at $\pm p_1(\lambda)$.*

PROOF: Recall that for case $\lambda = 0$ every point in $J(F_0)$ has exactly two preimages, i.e., the map F_0 is 2-to-1. The disk D_0 is mapped 2-to-1 onto the disk D_1, as no other regions are mapped onto D_1. Define D_2 to be the disk symmetrically located to D_1 with respect to the imaginary axis. Regions D_1 and D_2 are both mapped 1-to-1 onto D_0. By the boundary mapping principle ∂D_1 and ∂D_2 are both mapped 1-to-1 onto ∂D_0, and ∂D_0 is mapped 2-to-1 onto ∂D_1.

For the case $J(F_\lambda)$ where $\lambda \neq 0$, zero is mapped to infinity, thus there is a trap door T_λ about zero that is mapped to B_λ by F_λ. Under F_λ, every point within bulb D_0 is mapped within bulb D_1. The map F_λ is continuous, so by the boundary mapping principle ∂D_0 must be mapped onto ∂D_1. We know that $F_\lambda(-1) \in B_\lambda$, so ∂T_λ must be mapped 2-to-1 onto the outer boundary of D_1. The perturbed quadratic map is 4-to-1 and maps both the real and imaginary axis onto the real axis. Hence, we are able to locate the four preimages of $p_1(\lambda)$ along the real line. They are as follows:

1. The fixed point $p_1(\lambda)$ itself.

2. The point $-p_1(\lambda)$ is also a preimage. This can be shown by graphical analysis, or the symmetry of map F_λ and $J(F_\lambda)$ with respect to the imaginary axis.

3. The negative real point in $\partial T_\lambda \cap \mathbb{R}$ after removing the attachments. We know ∂T_λ is mapped to the 'outer' boundary of D_1 at ∂B_λ. The 4-to-1 map takes the real points of ∂T_λ onto either $\pm p_1(\lambda)$. There are two purely imaginary values in ∂T_λ that map onto the real axis. Due to the continuity of F_λ the image of the ∂T_λ wraps around the outer boundary of D_1 exactly two times. This implies that the two real values of $\partial D_0 \cap \partial T_\lambda$ are mapped to $p_1(\lambda)$. The two purely imaginary points of $\partial D_0 \cap \partial T_\lambda$ are mapped to the other real value on the outer boundary of D_1, and are mapped to $-p_1(\lambda)$ under F_λ^2.

4. The positive real point in $\partial T_\lambda \cap \mathbb{R}$ after removing the attachments. This can be readily verified using symmetry assumptions on T_λ and F_λ, or through graphical analysis.

We find that the four preimages of $p_1(\lambda)$ are all real numbers. Furthermore, the purely imaginary points of ∂D_0 map to $-p_1(\lambda)$ under F_λ^2. QED

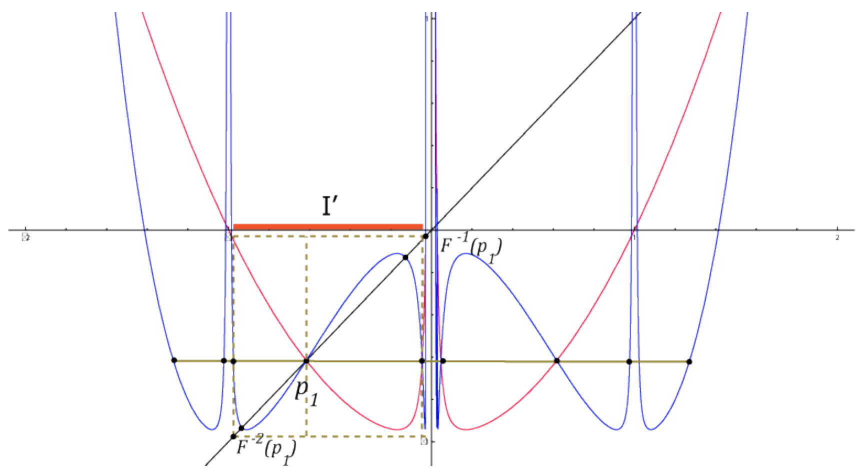


Figure 4: Graphical analysis of the first and second iterate of $F_\lambda(z)$. We illustrate the fixed point $p_1(\lambda)$ and its preimages under the first and second iterate. I' is the invariant interval. The dashed lines indicates regions from which orbits of points within I' cannot escape under F_λ^2.

Using Proposition 2, we complete the proof of the existence of an invariant interval connecting T_λ and $p_2(\lambda)$. We know that purely imaginary and purely real values of ∂D_0 are mapped onto $\mathbb{R} \cap \partial D_1$. Next, the real dynamics of the interval I' described in Corollary 1 is studied to prove the existence of an invariant interval under the F_λ^2.

Proposition 3 *There is an invariant interval I' in the filled Julia set under the map F_λ^2 connecting T_λ to the preimage of T_λ in D_1.*

PROOF: By the graphical analysis shown in Figure 4, under the second iterate all orbits of points between the first and second preimages of the fixed point $F_\lambda^{-1}(p_1)$ and $F_\lambda^{-2}(p_1)$ remain in interval I'. QED

The third iterate of the critical point is $-15/16$ and lies in the invariant interval I', so the critical orbit remains bounded for all iterations.

4 Construction of the Invariant Cantor Necklace

In this section, we prove the existence of a Cantor necklace in $J(F_\lambda)$ for small enough $\lambda \in \mathbb{R}^+$. The Cantor necklace will guarantee the convergence of $J(F_\lambda)$ to the filled basilica. Cantor necklaces have been recently introduced into the literature [5], and have proven useful for studying the dynamics for perturbed rational maps [3], [8].

Definition 1 *A Cantor necklace is a set homeomorphic to the following subset of the plane: construct a Cantor middle thirds set, and place an open disk of radius $1/(3^j \cdot 2)$ at the midpoint of each of the $1/3^j$ segments not contained in the Cantor middle thirds set with corresponding lengths. The Cantor necklace is the union of the Cantor middle thirds set and the family of open disks (see Figure 5).*

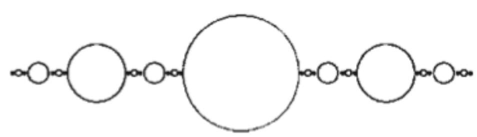


Figure 5: The Cantor middle thirds necklace.

Using symbolic dynamics we show the existence of a Cantor necklace in region D_0 of the dynamical plane. Let I_0 denote the fundamental sector in the disk D_0 containing $z = |z|e^{i\theta}$ with $0 < \theta < \pi/2$ between ∂T_λ and ∂B_λ. Similarly, let I_1 denote the sector with $\pi/2 < \theta < \pi$. Let $I_2 = -I_0$ and $I_3 = -I_1$ denote the remaining two fundamental sectors, see Figure 6.

Proposition 4 *D_0 can be divided into four disjoint fundamental sectors along real and imaginary axes such that under F_λ each fundamental sector is mapped 1-to-1 onto D_1. Under F_λ^2 each fundamental sector is mapped 1-to-1 onto D_0.*

PROOF: Proposition 2 characterized the images under F_λ of the purely real and imaginary points of ∂D_0 and ∂D_1. Using the continuity of F_λ, we invoke the boundary mapping principle to find the images of all boundary points of D_0 and D_1. The images of the interior of the fundamental sectors can be derived from the images of their respective boundary points. In Figure 6, we give the resulting images of the four fundamental sectors under F_λ^2, where boundaries are colored edges. The critical points c_λ are mapped to the critical values at $-1 \pm 2\sqrt{\lambda}$. The four preimages of T_λ must be contained in the Fatou set. QED

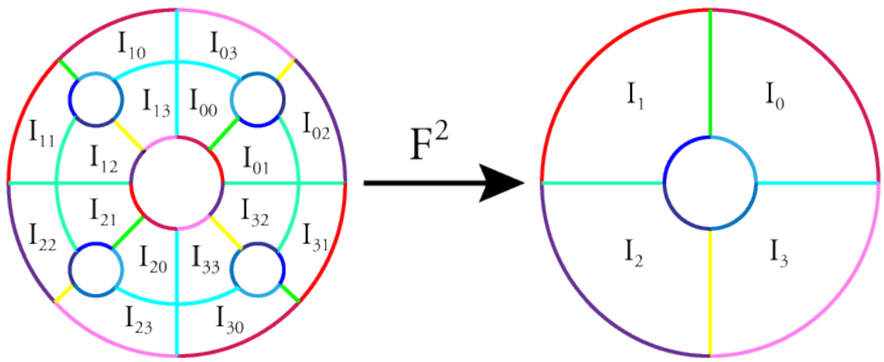


Figure 6: Preimages of the fundamental sectors in D_0. The four preimages of D_0 in $J(F_\lambda)$ under F_λ^2 are illustrated. Segments of the same color are mapped onto each other.

By Proposition 4, we have that $F_\lambda^2 : D_0 \to D_0$ is a 4-to-1 map. We assign indices $i_1 i_2 i_3 \dots$ to the subdivided parts of the fundamental sectors, such that $i_1 i_2 i_3 \dots \in \{s_1 s_2 \dots | s_j \in \{0,1,2,3\}\}$, in a way that the indices satisfy $F_\lambda^2(I_{i_1 i_2 i_3 \dots}) = I_{i_2 i_3 \dots}$; see Figure 6.

We construct a conjugacy between F_λ^2 and the shift map on two symbols $i_k \in \{0, 2\}$. It follows from Proposition 4 that F_λ^2 maps both I_0 and I_2 1-to-1 onto $D_0 \cup T_\lambda$. Let Γ_λ be the set of points which remain in $I_0 \cup I_2$ for all iterations. We have shown the following statement.

Proposition 5 *$F_\lambda^2 | \Gamma_\lambda$ is conjugate to the shift map on two symbols.*

Finally, we construct a Cantor necklace in the filled Julia set analogous to that in [2].

Theorem 1 *Γ_λ together with T_λ and the subset of preimages of B_λ remaining in $I_0 \cup I_2$ before landing on T_λ, form a Cantor necklace in the dynamical plane under F_λ^2.*

PROOF: Using Proposition 5 we construct the Cantor necklace in the dynamical plane. The central disk in the Cantor necklace is T_λ. The real points of ∂T_λ are sent to 1/3 and 2/3 in the Cantor necklace. The critical values lie in the filled Julia set, so the trap door has 4 preimages, of which by symmetry only two lie in the fundamental sectors $I_0 \cup I_2$. The boundary of these preimages each contain two preiamges of the real points of ∂T_λ, these points are junction points in Figure 6. The disks with diameter $1/3^n$ in the Cantor necklace correspond to the n-th preimeges of T_λ in regions I_0 and I_2, and the points in Γ_λ are mapped in a natural manner to the points in the Cantor middle thirds set. The first three steps of the construction are illustrated Figure 7. QED

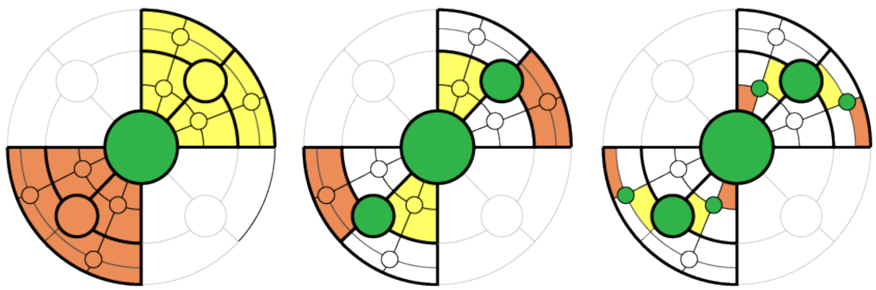


Figure 7: The first three steps in the construction of the Cantor necklace stretching over D_0. This is a homeomorphic generalization of the Cantor middle thirds necklace shown in Figure 5. The green regions are preimages of B_λ, the yellow and brown sectors trace the evolution of the Cantor necklace.

5 Proof of the Julia set converging to the filled Basilica

Using the Cantor necklace in the dynamical plane constructed in Theorem 1, we conclude the proof of the Julia set converging to the filled basilica. We state a theorem analogous to Theorem 3 in [3]. Adapting the proof for a topologically different Julia set we prove our result.

Theorem 2 *Let $B_\epsilon(z_0)$ denote the disk of radius $\epsilon > 0$ centered at z_0. There exists a $\mu > 0$ such that, for any $\lambda \in \mathbb{R}^+$ such that $0 < |\lambda| \leq \mu$, $J(F_\lambda) \cap B_\epsilon(z_0) \neq \emptyset$ for all $z_0 \in D_0$, i.e., the Julia set $J(F_\lambda)$ converges to the filled basilica.*

PROOF: Assume the theorem does not hold. For all $\epsilon > 0$, there is a sequence of $\lambda_j \to 0$ and a sequence of points $z_k \in D_0$ such that $J(F_{\lambda_j}) \cap B_{2\epsilon}(z_j) \neq \emptyset$ for all j. D_0 is a compact set, so there exists a subsequence z_j that converges to $z^* \in D_0$. For all parameter values in this subsequence $J(F_{\lambda_j}) \cap B_{2\epsilon}(z^*) \neq \emptyset$. Consider the circle of radius $|z^*|$ about the origin. Let $\gamma = \{z | \, |z| = z^*\} \cap B_\epsilon(z^*)$, and denote $l = |\gamma|$. Select k such that $2^{2k} l \geq 2\pi$. As $\lambda_j \to 0$, if j is large enough then $|F^i_{\lambda_j} - F^i_0|$ is small for $1 \leq i \leq k$. D_0 is invariant under F^2_λ, furthermore $F^{2k}_{\lambda_j}(\gamma)$ winds around the origin at least once. By Theorem 1, $F^{2k}_{\lambda_j}$ meets the Cantor necklace stretching across D_0, hence $J(F^{2k}_{\lambda_j}) \cap B_{2\epsilon}(z^*) \neq \emptyset$. By the backwards invariance of the Julia set under map F_λ, this result proves that the Julia set converges to the filled basilica as $\lambda \to 0+$. QED

References

[1] Blanchard, P., Devaney, R.L., Garijo, A., Marotta, S.M., Russel E. D. *Rabbits, Basilicas, and Other Julia Sets Wrapped in Sierpinski*

Carpets, Complex Dynamics: Families and Friends, A. K. Peters (2009), 277-296.

[2] Devaney, R.L., *Cantor Necklaces and Structurally Unstable Sierpinski Curve Julia Sets for Rational Maps*, Qualitative Theory of Dynamical Systems 5 (2006), 337-359.

[3] Devaney, R.L., Garijo, A. *Julia Sets Converging to the Unit Disc*, Proc. AMS, Vol. 136 (2008) 981-988.

[4] Devaney, R.L., Josic, K., Shapiro, Y. *Singular Perturbations of Quadratic Maps*, International Journal of Bifurcation and Chaos 14 (2004), 161-169.

[5] Devaney, R.L., Look, D. M., *Symbolic Dynamics for a Sierpinski Curve Julia Set*, Journal of Difference Equations and Applications. 11 (2005), 581-596.

[6] Morabito, M., Devaney, R.L., *Limiting Julia Sets for Singularly Perturbed Rational Maps*, International Journal of Bifurcation and Chaos 18 (2008), 3175-3181.

[7] McMullen, C., *Automorphisms of Rational Maps*, Holomorphic Functions and Moduli. Vol. 1. Math. Sci. Res. Inst. Publ. 10, Springer, New York (1988).

[8] Milnor, J., *Dynamics in One Complex Variable: Introductory Lectures*, 2nd Edition, Vieweg Verlag (2000).

About the author:

Robert Thijs Kozma

Robert is a math major student at Boston University (BU), College of Arts and Sciences (CAS). At the Dept. Mathematics and Statistics, Robert is doing research under the supervision of Professor Robert L. Devaney in the field of complex nonlinear dynamical systems. Robert received the Robert E. Bruce Prize for Excellence in Mathematics, the Undergraduate Research Opportunities Program Award, and is a CAS College Scholar for the 2009-2010 academic year. Robert also works as a laboratory assistant at the BU Dept. Cognitive Neural Systems. He is involved in DARPA research aimed at developing neuromorphic hardware on memristive nanodevices.
Department of Mathematics and Statistics, Boston University, 111 Cummington st, Boston MA 02215. rkozma@bu.edu