Julia Sets of Perturbed Quadratic Maps Converging to the Filled Basilica

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Abstract

In this paper we investigate singularly perturbed complex quadratic polynomial maps of the form

\[ F_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^2}. \]

We prove that for parameter values \( \lambda \in \mathbb{R}^+ \) as \( \lambda \to 0 \) the Julia set of \( F_\lambda(z) \) converges to the filled basilica when \( \lambda \neq 0 \).

1 Introduction

Our goal in this paper is to show that the Julia sets of the functions \( z^2 - 1 + \lambda/z^2 \) converge to the filled basilica, the filled Julia set of \( z^2 - 1 \), as \( \lambda \to 0 \) along the positive real axis. Julia sets of singularly perturbed polynomial maps from the family \( H_\lambda(z) = z^n + c + \lambda/z^d \) have been extensively studied in recent years. These maps are obtained by replacing the only finite critical point of \( H_0(z) = z^n \) by a pole of order \( n \).

For case \( c = 0 \), it is well known that the filled Julia set of \( H_0(z) = z^n \) is the unit disk [8]. The family of maps \( H_\lambda(z) = z^n + \lambda/z^d \) was investigated for \( d = 1 \) in [6], and for \( n, d \geq 2 \) in [3]. For family \( H_\lambda \) the Julia sets behave in three different ways as \( \lambda \to 0 \). In the case \( d = 1 \), the Julia set converges to the closed unit disk as \( \lambda \to 0 \) along \( n - 1 \) special rays in \( \mathbb{C} \). For \( n = d = 2 \) the Julia set converges to the unit disk as \( \lambda \to 0 \) from all directions in \( \mathbb{C} \). In cases with \( n, d > 2 \), regardless of how small \( |\lambda| \), Julia sets no longer converge, as there is always an annulus of fixed size about the origin in the complement of the Julia set.

In [1] the authors investigate the family \( H_\lambda(z) = z^n + c + \lambda/z^d \) for various \( c \) values located at centers of hyperbolic components of the Mandelbrot set. It is shown that the Julia set explodes for \( \lambda \neq 0 \), and for cases where \( c \neq 0 \) the boundaries of the components of the basin of infinity are not simple closed

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curves, but rather doubly inverted copies of the Julia set of $z^2 + c$. In this paper, we use the boundary structure of the Julia set to show that the Julia set of $F_\lambda(z) = z^2 - 1 + \lambda/z^2$ converges to the filled basilica.

This paper is structured as follows. In Section 2, we outline preliminaries on Julia sets. In Section 3, we show that the critical orbits of $F_\lambda$ never escape to infinity for small $|\lambda|$, by finding an invariant interval in the filled Julia set under the second iterate of $F_\lambda$ containing the critical orbit. The second iterate of $F_\lambda$ is denoted $F_\lambda^2$. In Section 4, we show that the dynamics of $F_\lambda^2$ on a portion of the Julia set is conjugate to the shift map on two symbols, and use symbolic dynamics to construct a Cantor necklace in the dynamical plane. In Section 5, we conclude the paper by proving that the existence of the invariant Cantor necklace in the dynamical plane implies that the Julia set converges to the filled basilica as $\lambda \to 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Filled Julia sets for maps (i) $F_{0.001}$; (ii) $F_{0.00001}$; (iii) $F_0$, called the filled basilica. Black points have bounded orbits, while colored points escape to infinity; the Julia set is the boundary of the colored region.}
\end{figure}

\section{Preliminaries}

The Julia set is the chaotic domain of a complex analytic function. Equivalently, the Julia set is the boundary of the basin of attraction of infinity, which for our maps is a superattracting fixed point. The complement of the Julia set is called the Fatou set. All attracting periodic orbits lie within the Fatou set, while the boundary of the Fatou set, as well as all repelling periodic points are contained in the Julia set. The Fatou set typically has simple dynamics, with most orbits converging to attracting cycles.

The filled Julia set consists of all points whose orbits are bounded. The Julia set denoted $J(F_\lambda)$ is the boundary of the filled Julia set. The Julia set of the unperturbed quadratic map $J(F_0)$ is called the basilica and is shown in Figure 1 (iii). We denote the immediate basin of attraction of infinity by $B_\lambda$. The region about the origin in the dynamical plane that is mapped to $B_\lambda$ under one iteration is called the trap door and is denoted $T_\lambda$. 

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It is known that $T_\lambda$ and $B_\lambda$ are disjoint when $|\lambda|$ is sufficiently small; in this paper we assume this to always be the case. The boundary of the immediate basin of attraction of infinity under $F_\lambda$ is denoted $\partial B_\lambda$ and is homeomorphic to $J(F_0)$. The map $F_\lambda$ restricted to $\partial B_\lambda$ is conjugate to $F_0$ on $J(F_0)$. However, the inner structure of $J(F_\lambda)$ is much more intricate: there are infinitely many doubly inverted copies of $J(F_0)$ for $\lambda$ values near zero [1]. Examples of doubly inverted basilicas are given in Figure 2.

3 Critical Orbit Remains Bounded

In this section, we prove necessary conditions for $J(F_\lambda)$ converging to the filled basilica as $\lambda \to 0^+$. The dynamical behavior of the critical points determine the structure of the Julia set. If the second iterates of the critical points lie in the trap door, then previous results would allow the possibility of an annuli in the Fatou set, which would not allow the Julia set to convegre to the filled basilica [3]. We show that the critical orbit remains bounded for all iterations of $F_\lambda$ by finding an invariant interval in the filled Julia set of $F_\lambda$ extending between $T_\lambda$ and its preimage under $F_\lambda^2$.

The four distinct critical points of $F_\lambda$ are $c_\lambda = \lambda^{1/4}$. In the case of $\lambda \in \mathbb{R}$ the four $c_\lambda$ are mapped to $-1 \pm 2\sqrt[4]{\lambda}$. By simple analysis, it follows that $\lim_{\lambda \to 0} F_\lambda^3(c_\lambda) = -15/16$. This value lies a constant distance from the preimage of $T_\lambda$ about $-1$ for $|\lambda|$ small enough.

We introduce the following notation to help in the discussion of Julia sets. Let $D_0$ denote the bulb of the Julia set of $F_0$ that contains 0. Similarly, let $D_1$ be the bulb of the Julia set about $-1$. The corresponding objects in the perturbed map $F_\lambda$ are denoted the same.

We prove that the filled Julia set of $F_\lambda$ contains a connected interval along the real axis running between $T_\lambda$ and its preimage in $D_1$. This is done by showing no preimages of $T_\lambda$ are located in this interval. The orbits of all points within this interval, including $-15/16$, remain bounded for all iterations of $F_\lambda$. 

Figure 2: The filled Julia set of $F_{0.00001}$ enlarged about the trap door, with details of the doubly inverted basilicas.
\textbf{Proposition 1} Let $p_2(\lambda)$ denote the negative fixed point of $F_\lambda(z)$ near zero. The interval $I' = [F_\lambda^{-1}(p_2(\lambda)), p_2(\lambda)]$ contains the local minima of the quadratic map, i.e., $F_\lambda^{-1}(p_2(\lambda)) < -1 + 2\sqrt{\lambda}$ for all $\lambda$.

\textsc{Proof:} We obtain an algebraically closed form for $p_2(\lambda)$ and its preimage $F_\lambda^{-1}(p_2(\lambda))$ in terms of $\lambda$ in 4 steps:

1. In order to find the fixed point, we solve the quartic equation $z^2 - 1 + \lambda/z^2 = z$ using Ferrari’s method. Multiplying by $z^2$ and rearranging terms we obtain $z^4 - z^3 - z^2 + \lambda = 0$. Using Ferrari’s method we transform our quartic into the cubic $y^3 + y^2 + 4\lambda y - 5\lambda = 0$.

2. We use Cardano’s formula to solve the cubic and obtain a formula of the fixed point $p_2(\lambda)$ in terms of $\lambda$. The details are left to the reader.

3. We substitute the value of the fixed point $p_2(\lambda)$ into the biquadratic formula for the preimage of $F_\lambda$. Hence we obtain a closed form equation for $F_\lambda^{-1}(p_2(\lambda))$ with $\lambda$ as the only parameter.

$$F_\lambda^{-1}(\lambda) = -\sqrt{\frac{p_2(\lambda) + 1 + \sqrt{-(p_2(\lambda) + 1)^2 - 4\lambda}}{2}}$$

\begin{equation}
(1)
\end{equation}

4. For small $|\lambda|$ values, we have that $F_\lambda^{-1}(p_2(\lambda)) < -1 + 2\sqrt{\lambda}$. Hence the local minimum of the perturbed quadratic map is contained in the interval $I' = [F_\lambda^{-1}(p_2(\lambda)), p_2(\lambda)]$ completing the proof.
Note that solving with Ferrari’s method and Cardano’s formula require no numerical methods so the proof of Proposition 1 is mathematically rigorous.

Let \( p_1 = (1 - \sqrt{5})/2 \) denote the fixed point of the unperturbed quadratic map. Let \( I = [F^{-2}(p_1), F^{-1}(p_1)] \). In this case \( I \subseteq I' \), see Figure 3.

**Corollary 1** As \( \lambda \to 0 \), the closed interval \( I \) contains one of the two local minima of \( F_\lambda \) along the real axis.

By graphical analysis of \( F_\lambda \) over the interval \( I \) given in Figure 3, we find that the interval \( I \) is invariant under \( F_\lambda \).

**Corollary 2** \( F_\lambda \) maps the closed interval \( I \) 2-to-1 onto itself.

In the next proposition, we show the intervals \( I \) and \( I' \) both contain the segment connecting the \( T_\lambda \) to its preimage, by proving that \( p_2(\lambda) \) lies on \( \partial T_\lambda \), and that the \( I' \) interval is invariant under \( F_\lambda^2 \). As a result there is a connected piece in \( J(F_\lambda) \) joining \( T_\lambda \) and its preimage in \( D_1 \). Moreover, we find that \( p_2(\lambda) > F^{-1}(p_1) \) where \( p_2(\lambda) \) is the negative fixed point near 0. One can find a closed form for the fixed point \( p_1(\lambda) \), the fixed point near \( p_1 = (1 - \sqrt{5})/2 \), in terms of \( \lambda \) using the method outlined for \( p_2(\lambda) \) in Proposition 1.

**Proposition 2** Two preimages of the fixed point \( p_1(\lambda) \) lie at \( \partial T_\lambda \cap \mathbb{R} \), and two lie at \( \pm p_1(\lambda) \).

**Proof:** Recall that for case \( \lambda = 0 \) every point in \( J(F_0) \) has exactly two preimages, i.e., the map \( F_0 \) is 2-to-1. The disk \( D_0 \) is mapped 2-to-1 onto the disk \( D_1 \), as no other regions are mapped onto \( D_1 \). Define \( D_2 \) to be the disk symmetrically located to \( D_1 \) with respect to the imaginary axis. Regions \( D_1 \) and \( D_2 \) are both mapped 1-to-1 onto \( D_0 \). By the boundary mapping principle \( \partial D_1 \) and \( \partial D_2 \) are both mapped 1-to-1 onto \( \partial D_0 \), and \( \partial D_0 \) is mapped 2-to-1 onto \( \partial D_1 \).

For the case \( J(F_\lambda) \) where \( \lambda \neq 0 \), zero is mapped to infinity, thus there is a trap door \( T_\lambda \) about zero that is mapped to \( B_\lambda \) by \( F_\lambda \). Under \( F_\lambda \), every point within bulb \( D_0 \) is mapped within bulb \( D_1 \). The map \( F_\lambda \) is continuous, so by the boundary mapping principle \( \partial D_0 \) must be mapped onto \( \partial D_1 \). We know that \( F_\lambda(-1) \in B_\lambda \), so \( \partial T_\lambda \) must be mapped 2-to-1 onto the outer boundary of \( D_1 \). The perturbed quadratic map is 4-to-1 and maps both the real and imaginary axis onto the real axis. Hence, we are able to locate the four preimages of \( p_1(\lambda) \) along the real line. They are as follows:

1. The fixed point \( p_1(\lambda) \) itself.
2. The point \( -p_1(\lambda) \) is also a preimage. This can be shown by graphical analysis, or the symmetry of map \( F_\lambda \) and \( J(F_\lambda) \) with respect to the imaginary axis.

QED
3. The negative real point in $\partial T_\lambda \cap \mathbb{R}$ after removing the attachments. We know $\partial T_\lambda$ is mapped to the 'outer' boundary of $D_1$ at $\partial B_\lambda$. The 4-to-1 map takes the real points of $\partial T_\lambda$ onto either $\pm p_1(\lambda)$. There are two purely imaginary values in $\partial T_\lambda$ that map onto the real axis. Due to the continuity of $F_\lambda$ the image of the $\partial T_\lambda$ wraps around the outer boundary of $D_1$ exactly two times. This implies that the two real values of $\partial D_0 \cap \partial T_\lambda$ are mapped to $p_1(\lambda)$. The two purely imaginary points of $\partial D_0 \cap \partial T_\lambda$ are mapped to the other real value on the outer boundary of $D_1$, and are mapped to $-p_1(\lambda)$ under $F_\lambda^2$.

4. The positive real point in $\partial T_\lambda \cap \mathbb{R}$ after removing the attachments. This can be readily verified using symmetry assumptions on $T_\lambda$ and $F_\lambda$, or through graphical analysis.

We find that the four preimages of $p_1(\lambda)$ are all real numbers. Furthermore, the purely imaginary points of $\partial D_0$ map to $-p_1(\lambda)$ under $F_\lambda^2$. QED

Figure 4: Graphical analysis of the first and second iterate of $F_\lambda(z)$. We illustrate the fixed point $p_1(\lambda)$ and its preimages under the first and second iterate. $I'$ is the invariant interval. The dashed lines indicates regions from which orbits of points within $I'$ cannot escape under $F_\lambda^2$.

Using Proposition 2, we complete the proof of the existence of an invariant interval connecting $T_\lambda$ and $p_2(\lambda)$. We know that purely imaginary and purely real values of $\partial D_0$ are mapped onto $\mathbb{R} \cap \partial D_1$. Next, the real dynamics of the interval $I'$ described in Corollary 1 is studied to prove the existence of an invariant interval under the $F_\lambda^2$.

**Proposition 3** There is an invariant interval $I'$ in the filled Julia set under the map $F_\lambda^2$ connecting $T_\lambda$ to the preimage of $T_\lambda$ in $D_1$. 

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Proof: By the graphical analysis shown in Figure 4, under the second iterate all orbits of points between the first and second preimages of the fixed point $F^{-1}_\lambda(p_1)$ and $F^{-2}_\lambda(p_1)$ remain in interval $I'$. QED

The third iterate of the critical point is $-15/16$ and lies in the invariant interval $I'$, so the critical orbit remains bounded for all iterations.

4 Construction of the Invariant Cantor Necklace

In this section, we prove the existence of a Cantor necklace in $J(F_\lambda)$ for small enough $\lambda \in \mathbb{R}^+$. The Cantor necklace will guarantee the convergence of $J(F_\lambda)$ to the filled basilica. Cantor necklaces have been recently introduced into the literature [5], and have proven useful for studying the dynamics for perturbed rational maps [3], [8].

Definition 1 A Cantor necklace is a set homeomorphic to the following subset of the plane: construct a Cantor middle thirds set, and place an open disk of radius $1/(3^j \cdot 2)$ at the midpoint of each of the $1/3^j$ segments not contained in the Cantor middle thirds set with corresponding lengths. The Cantor necklace is the union of the Cantor middle thirds set and the family of open disks (see Figure 5).

![Figure 5: The Cantor middle thirds necklace.](image)

Using symbolic dynamics we show the existence of a Cantor necklace in region $D_0$ of the dynamical plane. Let $I_0$ denote the fundamental sector in the disk $D_0$ containing $z = |z|e^{i\theta}$ with $0 < \theta < \pi/2$ between $\partial D_0$ and $\partial D_1$. Similarly, let $I_1$ denote the sector with $\pi/2 < \theta < \pi$. Let $I_2 = -I_0$ and $I_3 = -I_1$ denote the remaining two fundamental sectors, see Figure 6.

Proposition 4 $D_0$ can be divided into four disjoint fundamental sectors along real and imaginary axes such that under $F_\lambda$ each fundamental sector is mapped 1-to-1 onto $D_1$. Under $F_\lambda^2$ each fundamental sector is mapped 1-to-1 onto $D_0$.

Proof: Proposition 2 characterized the images under $F_\lambda$ of the purely real and imaginary points of $\partial D_0$ and $\partial D_1$. Using the continuity of $F_\lambda$, we invoke the boundary mapping principle to find the images of all boundary points of $D_0$ and $D_1$. The images of the interior of the fundamental sectors can be derived from the images of their respective boundary points. In Figure 6, we give the resulting images of the four fundamental sectors under $F_\lambda^2$, where boundaries
are colored edges. The critical points $c_\lambda$ are mapped to the critical values at $-1 \pm 2\sqrt{\lambda}$. The four preimages of $T_\lambda$ must be contained in the Fatou set. QED

By Proposition 4, we have that $F^2_\lambda : D_0 \to D_0$ is a 4-to-1 map. We assign indices $i_1i_2i_3\ldots$ to the subdivided parts of the fundamental sectors, such that $i_1i_2i_3\ldots \in \{s_1s_2\ldots | s_j \in \{0,1,2,3\}\}$, in a way that the indices satisfy $F^2_\lambda(I_{i_1i_2i_3\ldots}) = I_{i_2i_3\ldots}$; see Figure 6.

We construct a conjugacy between $F^2_\lambda$ and the shift map on two symbols $i_k \in \{0,2\}$. It follows from Proposition 4 that $F^2_\lambda$ maps both $I_0$ and $I_2$ 1-to-1 onto $D_0 \cup T_\lambda$. Let $\Gamma_\lambda$ be the set of points which remain in $I_0 \cup I_2$ for all iterations. We have shown the following statement.

**Proposition 5** $F^2_\lambda|\Gamma_\lambda$ is conjugate to the shift map on two symbols.

Finally, we construct a Cantor necklace in the filled Julia set analogous to that in [2].

**Theorem 1** $\Gamma_\lambda$ together with $T_\lambda$ and the subset of preimages of $B_\lambda$ remaining in $I_0 \cup I_2$ before landing on $T_\lambda$, form a Cantor necklace in the dynamical plane under $F^2_\lambda$.

**Proof:** Using Proposition 5 we construct the Cantor necklace in the dynamical plane. The central disk in the Cantor necklace is $T_\lambda$. The real points of $\partial T_\lambda$ are sent to $1/3$ and $2/3$ in the Cantor necklace. The critical values lie in the filled Julia set, so the trap door has 4 preimages, of which by symmetry only two lie in the fundamental sectors $I_0 \cup I_2$. The boundary of these preimages each contain two preimages of the real points of $\partial T_\lambda$, these points are junction points in Figure 6. The disks with diameter $1/3^n$ in the Cantor necklace correspond to the $n$-th preimages of $T_\lambda$ in regions $I_0$ and $I_2$, and the points in $\Gamma_\lambda$ are mapped in a natural manner to the points in the Cantor middle thirds set. The first three steps of the construction are illustrated Figure 7. QED
Figure 7: The first three steps in the construction of the Cantor necklace stretching over $D_0$. This is a homeomorphic generalization of the Cantor middle thirds necklace shown in Figure 5. The green regions are preimages of $B_\lambda$, the yellow and brown sectors trace the evolution of the Cantor necklace.

5 Proof of the Julia set converging to the filled Basilica

Using the Cantor necklace in the dynamical plane constructed in Theorem 1, we conclude the proof of the Julia set converging to the filled basilica. We state a theorem analogous to Theorem 3 in [3]. Adapting the proof for a topologically different Julia set we prove our result.

**Theorem 2** Let $B_\epsilon(z_0)$ denote the disk of radius $\epsilon > 0$ centered at $z_0$. There exists a $\mu > 0$ such that, for any $\lambda \in \mathbb{R}^+$ such that $0 < |\lambda| \leq \mu$, $J(F_\lambda) \cap B_\epsilon(z_0) \neq \emptyset$ for all $z_0 \in D_0$, i.e., the Julia set $J(F_\lambda)$ converges to the filled basilica.

**Proof:** Assume the theorem does not hold. For all $\epsilon > 0$, there is a sequence of $\lambda_j \to 0$ and a sequence of points $z_k \in D_0$ such that $J(F_{\lambda_j}) \cap B_{2\epsilon}(z_k) \neq \emptyset$ for all $j$. $D_0$ is a compact set, so there exists a subsequence $z_j$ that converges to $z^* \in D_0$. For all parameter values in this subsequence $J(F_{\lambda_j}) \cap B_{2\epsilon}(z^*) \neq \emptyset$. Consider the circle of radius $|z^*|$ about the origin. Let $\gamma = \{z \mid |z| = z^*\} \cap B_\epsilon(z^*)$, and denote $l = |\gamma|$. Select $k$ such that $2^{2k}l \geq 2\pi$. As $\lambda_j \to 0$, if $j$ is large enough then $|F_{\lambda_j}^i - F_{\lambda_j}^0|$ is small for $1 \leq i \leq k$. $D_0$ is invariant under $F_{\lambda_j}^2$, furthermore $F_{\lambda_j}^{2k}(\gamma)$ winds around the origin at least once. By Theorem 1, $F_{\lambda_j}^{2k}$ meets the Cantor necklace stretching across $D_0$, hence $J(F_{\lambda_j}^{2k}) \cap B_{2\epsilon}(z^*) \neq \emptyset$. By the backwards invariance of the Julia set under map $F_\lambda$, this result proves that the Julia set converges to the filled basilica as $\lambda \to 0+$. QED

References

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